In the midst of this political turmoil the name of the astronomer Jan Šindel[1] (b. 1375) stands out. He is of interest to Italians as the friend of Giovanni Bianchi[2] and a man greatly esteemed by Enea Silvio, afterwards Pope Pius II. Even a man like Tyge Brahe[3] studied the works of Šindel.

I do not wish to try the patience of the reader with the names of all the mathematicians of minor importance who, however, prepared the field for scientific study of the stars; for the most part these were called from foreign countries to the court of Rudolph II.

At the end of the fifteenth century a monument was set up which showed the astronomical attainment of the Czechs at that time, namely the Clock of Prague whose maker was called, according to tradition, Master Hanus. Not only the movable statues of the clock diverted the public, but its construction won the admiration of the mathematical astronomers of the fifteenth and sixteenth centuries. The clock showed the motion, the rising and setting of the stars, the sun, moon, and planets, the phases of the moon, the height and depth of the sun and moon above and below the horizon, the calendar of feast days, the hours according to the ancient numeration, that is from one to twenty-four commencing with sunset, and finally the astrological rulers of every day and every hour.[4]

[1] This name is usually spelled Szindel (other forms: Syndelius and Schinttel). Jan Szindel (in German, Johann Schindel) was born in Králové Hradec (Königgrätz), Bohemia, not earlier than 1370 nor later than 1375; it is not known that he was born in 1375. He died in Prag about 1450. After being director of the St. Niclas School in Prague, he was dozent in mathematics and astronomy in Vienna, 1407–1409. From 1410 on, he was "doctor et lector ordinarius Universitatis studii Pragensis." The following sources may be consulted for material treating of Szindel's life and work: L. J. Scherschnik, *De doctis Regiopradecensibus commentarius*, Prague, 1775; Truhlář, *Počátky humanismu v Čechšch* (The beginning of humanism in Bohemia), 1892, pp. 24–27; K. Teige, "New information completing the history of mathematical science. 2. Jan Sindel" (in Bohemian), *Časopis pro pěst. matem.*, vol. 22, 1893, pp. 244–246; A. Czerny, "Aus dem Briefwechsel des grossen Astronomen Georg von Peurbach," *Archiv für österreichische Geschichte*, vol. 72, 1888, pp. 299–301; J. Teige, "Ein Beitrag zur Lebensgeschichte des Magister Joannes de Praga," *Zeitschrift für Mathematik und Physik*, hist.-literar. Abt., vol. 28, pp. 41-44; F. J. Studnička, *Sitzungsberichte der kgl. böhm. Gesellschaft der Wissenschaften*, Math.-Naturwiss. 1892, pp. 103–104; S. Günther, *Geschichte des math. Unterrichts im deutschen Mittelalter bis Zum Jahre 1525*, Berlin, 1887, p. 228; Schepfs, "Der Mathematiker Joh. Schindel (1444)," *Anzeiger für Kunde, deutschen Vorzeit*, 1879, new series, vol. 26, col. 262 (see also 1878, vol. 25, col. 1); and J. P. Kalina von Jätenstein, *Nachrichten über böhmische Schriftsteller und Gelehrte*, Prague, 1818.

[2] It seems certain that this name should be Giovanni Bianchini, that of the fifteenth century astronomer of Ferrara, and the correspondent of Regiomontanus. Compare P. Riccardi, *Biblioteca Matematica Italiana*, Modena, 1893, (I), cols. 133–134; (II), cols. 7, 94–95; R. Wolf, *Geschichte der Astronomie*, Munich, 1877; A. R. Grant, *History of Physical Astronomy*, London, 1852; S. Günther, *Geschichte des mathematischen Unterrichts im deutschen Mittelalter bis zum Jahre 1525*, Berlin, 1887, p. 225; and M. Cantor, *Vorlesungen über Geschichte der Mathematik*, Leipzig, vol. 2, second edition, 1900, various references in the index.

[3] Tyge Ottesen Brahe (1546–1601) is popularly known as Tycho Brahe. For valuable bibliographical and historical notes concerning him see N. Nielsen, *Matematiken I Danmark 1528–1800* Copenhagen, 1912, pp. 27–37.

[4] This clock, one of the great marvels of Bohemia, is situated in the outer wall at the foot of the tower in the old Town Hall, on the market place, and was constructed in 1490 by Master, Hanus, an astronomer and mathematician who taught at the University. The old dial plate is protected by a Gothic canopy with ornamental conventional forms of leaves and animals, motions

In the sixteenth and at the beginning of the seventeenth century some manuals of arithmetic and mensuration were published. Their purpose was practical. The material was limited to the calculations of lines and figures, to the simple and complex rule of three and to similar problems, to arithmetic, and to geometric progressions.

Above the common level rise two astronomers, Kerásek Lvovický z Lvovicz (1514–1574)[1] and more especially Tadeáš Hájek z Hájek (1525–1600):[2] the first as author of several astronomical books, the other as professor of mathematics at the University and then as doctor of medicine to the Emperor Maximilian II. We know a group of his astronomical books, an *Oratio de laudibus geometriæ*,[3] and besides this, calendars, and minutiæ, and finally several books on medicine and chemistry. His greatest fame came through his fine description of the new star of the year 1572;[4] it is this star which is connected so closely with the scientific career of Tyge Brahe.[5] The great Danish astronomer considered the

of the heavenly bodies, etc. Above the dial are two windows which open as an hour is struck, and figures of Jesus followed by the apostles appear. Under the dial is another recent one in an ancient border representing the signs of the zodiac; it contains scenes depicting seasonal events in Bohemia (fishing time, hunting season, vintage, etc.). In the time of Hanus this calendar was an integral part of the clock; it was removed to the shelter of the museum of the Rudolphinum in the nineteenth century, and replaced by the above-mentioned modern copy by the painter Josef Manes.

Master Hanus had charge of the Prague clock until his death; then a pupil held the office till his death in 1530. He was followed by a third keeper who served for forty years and left a work on 15 strips of parchment describing the origin and mechanism of the clock. After his death the clock stopped, no one being found able to care for it till 1787. It stopped again in 1824 and remained out of commission till 1866.

An excellent picture of the clock is given in F. M. Feldhaus, *Die Technik der Vorzeit*, Leipzig, 1914. Another good picture is given in the article "orloj" (from which most of the information given above is taken) in *Ottův Slovník Naučný* (Slovak Cyclopædia of Knowledge), Prague, vol. 18, 1902. See also L. P. M. Leger, *Prague* (Les villes d'art célèbres), Paris, 1907, pp. 93–94; F. H. H. Lützow, *The Story of Prague*, London, second edition, 1907, pp. 80, 168. There is also a fine picture in C. Wakefield, *Visit to the City of Prague*, London, 1921, p. 44.

[1] Compare "Der Astronom Cyprianus Leovitius (1514–1574) und seine Schriften" by Jos. Mayer, *Bibliotheca Mathematica*, series 3, vol. 4, pp. 134–159, 1903. Reference may also be given to: C. F. Dechales Milliet, *Cursus seu mundus mathematicus*, Leyden, 1690, vol. 1, p. 88; P. Bayle, *Dictionnaire historique et critique*, Rotterdam, 1696, third edition 1720, etc.; J. F. Weidler, *Historia astronomiae, sive de ortu et progressu astronomiae*, Wittenberg, 1741, p. 369; C. G. Jöcher, *Allgemeines Gelehrten-Lexicon*, Leipzig, vol. 2, 1750; J. S. Bailly, *Histoire de l'astronomie moderne*, new edition, Paris, 1785, vol. 2, p. 236; A. G. Kästner, *Geschichte der Mathematik*, Göttingen, vol. 2, 1797, pp. 344, 538; R. Wolf, *Geschichte der Astronomie*, Munich, 1877, p. 303.

[2] Reference may be given to: F. M. Pelzel, *Abbildungen böhmischer und mährischer Gelehrten und Künstler*, vol. 4, Prague, 1782; J. S. Bailly, *Histoire de l'astronomie moderne*, new edition, Paris, 1785, vol. 1, pp. 375, 411. There is a brief paragraph concerning this scientist in Poggendorff, *Biographisch-literarisches Handwörterbuch*, vol. 1; this seems to have been the source of information for the paragraph on Hájek in D. E. Smith, "Medicine and mathematics in the sixteenth century," *Annals of Medical History*, New York, 1917, p. 136.

[3] Published at Prague in 1557.

[4] *De investigatione loci stellæ novæ in Zodiaco*, Vienna, 1573; *Dialexis de novæ et prius incognitæ stellæ inusitatæ magnitudinis apparitione et de ejusdem stellæ vero loco constituendo . . .*, Frankfurt, 1574.

[5] T. Brahe, *De nova et nvllivs. ævi memoria prius visa stella, . . . Anno . . . 1572. mense Novembrj primùm conspecta, contemplatio mathematica*, Copenhagen, 1573. See also *Tychonis Brahe Dani Opera Omnia* edidit I. L. E. Dreyer, vol. 1, Copenhagen, 1913.

observations of Hájek the best after his own. It is interesting that Hájek writes against astrological imposture. He believes, with his time, that comets and phenomena of the universe have an importance in the life of men, but, in his deeply religious nature, especially during his later years, he sees in them certain admonitions of God for the sins of men. He says expressly that it is not possible to predict the modes of divine punishments and that the omnipotent God can mitigate the punishment of a penitent world. And again we come to what I have already said. Hájek was greatly concerned with religious questions, he even maintained a polemic against an attack on the Bohemian Brothers though not of their religion. A hindrance to his astronomical activities was the lack of accurate instruments such as Tyge had. Not finding a patron like the beneficent Frederick II of Denmark, he had to make a living from his medical practice and gave only his spare time to science. Hájek was in active correspondence with the mathematical societies of his time. He continued his relations with the Italian scientists he had known while studying at Bologna; it was for this reason he visited Girolamo Cardano at Milan.[1] That he had an intimate friendship with Tyge Brahe, we see from their copious correspondence, a part of which remains still unpublished in the Bibliothek Nazional in Vienna. To him belongs the principal credit of calling Tyge Brahe to Prague. Hájek and Martin Bacháček, (1539–1612), professor of mathematics in the University, were the principal representatives of that "native" spirit of which I have already spoken.

The sojourns of Brahe, Kepler, and Bürgi[2] would have had a profound and beneficial influence on the development of the sciences in Bohemia, if there had not occurred a long prepared catastrophe. Scarcely had the Hapsburg dynasty got possession, in 1526, of the Bohemian crown, for which during two centuries and a half it had worked with a tenacity peculiar to itself, when it sought means completely to subjugate the hated Czech nation. The unfortunate battle of

[1] Cardano (1501(?)–1576) was a practising physician and professor of mathematics in Milan 1534–1559. It was in his *Artis magnae sive de regulis Algebrae Liber unus*, Milan, 1545, that he published the rule given him by Tartaglia under a pledge of secrecy six years earlier.

[2] Tyge Brahe (1546–1601) went to Prague in 1599 and died there. Johann Kepler (1571–1630) was in Prague from the time of Brahe's death to 1615. Joost Bürgi (1552–1632), chief clock-maker to Rudolph II, 1603–1622, is entitled to the honor of independent invention of logarithms and constructed the tables, *Arithmetische und Geometrische Progress-Tabulen*, published in 1620. Napier was working on logarithms at least as early as 1594; compare "Question of priority" by F. Cajori in *Napier Tercentenary Memorial Volume*, 1915. See also R. Wolf, *Johann Kepler und Jost (sic) Bürgi*, Zürich, 1872; and J. v. Hasner, *Tycho Brahe und J. Kepler im Prag. Eine Studie*, Prague, 1872. Tyge Brahe's tomb in the Tyn Church, near the University and Town Hall, shows an effigy in red marble representing Brahe in armor holding his globe and compasses. A photograph of this is reproduced in L. P. M. Leger, *Prague* (Les villes d'art célèbres), Paris, 1907, p. 9. See also F. H. H. V. Lützow, *The Story of Prague*, London, second edition, 1907, p. 104.

Among the publications which appeared on the three hundredth anniversary of T. Brahe's death were the following: (*a*) "Tyge Brahe in Bohemian literature" (in Bohemian), by L. Peprný, *Časopis ¦pro pěst. matem.*, vol. 30, 1901, pp. 209–222—a list, with indications of contents, of Czech publications discussing Brahe and his work; (*b*) *Prager Tychoniana zur bevorstehenden Säkularfeier der Erinnerung an das vor 300 Jahren erfolgte Ableben des Reformators der beobachtenden Astronomie Tycho Brahe gesammelt*, by F. J. Studníčka, Prague, 1901; 71 pp. + chromo-portrait + 7 reproductions—concerning some of Brahe's books, manuscripts, instruments, etc., now in Prague.

White Mountain in 1620 gave the desired pretext to inflict upon the whole Czech nation unheard-of cruelty. After the thirty years war there did not remain in Bohemia more than one-sixth of the old population; property of the native people was confiscated, and the finest and most intellectual Czechs wandered like beggars in foreign lands. Of course the development of the sciences would thus be interrupted and the participation of Czechs in international sciences greatly diminished.

The Czech Jan Marcus-Marci (1595–1667)[1] is worthy to be remembered. His writings on the squaring of the circle[2] and on physics, we know; he was the forerunner of Newton in the theory of the refraction of light.

The sciences in the seventeenth and eighteenth centuries found an asylum in the monasteries, especially among the Jesuits who took possession of the University of Prague. Among the native Jesuit Czechs were several mathematicians. The highest praise was given to Father Jakob Kreza (1648–1715)[3] who has left several mathematical works. Kreza, after having been professor in the Universities of Prague and Olomauc, was called to the University of Madrid, where he taught for fifteen years.

The eighteenth century produced many practical books, such as interest-tables and books of applied geometry. Of the practical manuals I would cite especially the one of Václav Veselý, to whom also is attributed an *Ars liberandi*, published in the year 1734.

Technical arts flourished in Bohemia. Already at the beginning of the eighteenth century a school of engineers was founded at Prague, not by the imperial government, but by the Diet of the Bohemian Kingdom. The Austrian government, relentlessly hostile to the Czech nation, wished to Germanize not only the offices but all the schools from the elementary to the university; for this reason Emperor Josef II excluded the Latin language at the end of the

[1] His life and work have been described by F. J. Studnička in a "Festvortrag," "Joannes Marcus Marci a Cronland, sein Leben und gelehrtes Wirken," *Jahresbericht der königl. böhm. Gesellschaft der Wissenschaften*, 1890, 32 pages. He is referred to as the "Bohemian Galileo" and a list is given of 16 of his works dealing with philosophy, geometry, physics, astronomy, and medicine. See also Marci a Kronland, *Liturgia mentis*, edited by J. J. W. Dobrzensky, Regensburg, 1678; F. M. Pelzel, *Abbildungen böhmischer und mährischer Gelehrten und Künstler*, Prague, vol. 1, 1773; F. Hoefer, *Nouvelle biographie générale*, Paris, vol. 28, 1859; V. Láska, "Ueber Marcus Marci de Kronland," *Zeitschrift für Mathematik und Physik* (hist. liter. Abth.), vol. 35, 1890, pp. 1–3; Poggendorff, *Biographisch-literarisches Handwörterbuch*, vol. 2; D. E. Smith, "Medicine and mathematics in the sixteenth century," *Annals of Medical History*, 1917, p. 136; and *Enciclopedia Universel Illustrada Evropeo-Americana*, vol. 33, Barcelona, 1920.

[2] *Labyrinthus, in quo via ad circuli quadraturam pluribus modus exhibetur*, Prague, 1654. Compare the reference to Marci by Huygens, *Oeuvres Complètes de Christian Huygens*, vol. 12, 1910, pp. 97–98.

[3] There is a brief paragraph concerning "Jakob Kresa" in Poggendorff, *Biographisch-literarisches Handwörterbuch*, vol. 1. The name is spelled as in the text in M. Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. 3, second edition, 1901, p. 12. It is here stated that while Kreza was in Madrid he translated into Spanish Euclid's Elements, books 1–6, 11, 12, which were published at Brussels in 1688; compare *Bibliotheca Mathematica*, series 3, vol. 12, p. 261, and P. Riccardi, *Saggio di una Bibliografia Euclidea*, Bologna, 1887, p. 50 of the first part. See A. Jemelka, ["On the life and works of the mathematician P. J. Kresa, S. J."] (in Bohemian), *Časopis pro pěst. matem.*, vol. 42, 1913, pp. 501–509.

eighteenth century and substituted for it the German language. A few years after the founding of the famous polytechnic school in Paris, the school of engineers in Prague was also changed by the Diet of the Bohemian Kingdom to a Polytechnic Institute of the type of the one in Paris. This was in 1806, that is, before the opening of the Polytechnic Institute in Vienna (1815).

In the eighteenth century, the leaders in the study of mathematics were Josef Stepling (1716–1778), author of many books on mathematics, astronomy and physics,[1] and Jan Tesánek (1728–1788)[2] who wrote a series of mathematical works, the commentaries on the celebrated work of Newton being noteworthy.[3]

At the end of the eighteenth century and at the beginning of the nineteenth there swept over Europe a wave of nationalism awakened by the grand ideas of the French revolution. Among the Czechs of the cultured class were idealists, who went to the huts of the peasants seeking the despised native language. They gave new life to the forgotten Czech literature, and in this way aroused national consciousness.

In the ranks of these idealists was a mathematician, professor at the University of Prague, Father Stanislav Vydra (1741–1804).[4] He wrote several mathematical books in Latin and German. In order to show that the Czechs had mathematical literature, he published in the year 1778 a *Historia matheseos in Bohemia et Moravia cultae*,[5] and in order to show that the language of the common people could serve also as a medium for the sciences, he wrote in Czech *Počátkové arithmetiky* (Elements of Arithmetic) and an algebra which however was left incomplete.

In the nineteenth century the Czechs had the great task of establishing modern Czech mathematical literature, creating an adequate terminology and

[1] Listed in Poggendorff, *Biographisch-literarisches Handwörterbuch*, vol. 2. A life of Stepling by S. Vydra was published at Prague in 1779: *Vita ad modum reverendi ac magnifici viri J. Stepling.* See also F. M. Pelzel, *Abbildungen böhmischer und mährischer Gelehrten und Künstler*, Prague, vol. 1, 1773; S. Vydra, *Historia matheseos in Bohemia et Moravia cultæ*, Prague, 1778; F. M. Pelzel, *Böhmische, mährische, und schlesische Gelehrte und Schriftsteller aus dem Order der Jesuiten*, Prague, 1786; J. G. Meusel, *Lexikon der von 1750 bis 1800 verstorbenen deutschen Schriftsteller*, Leipzig, vol. 13, 1815; *Oesterreichsche National-Encyklopädie*, Vienna, vol. 5, 1836.

[2] A common spelling of this name seems to be Tessánek. For authorities concerning his life, the works of Vydra, Pelzel (2), and Meusel, mentioned in the previous footnote, may be consulted. See also *Abhandlungen der königl. böhm. Gesellsch. d. Wiss.*, new series, vol. 4, 1837; F. J. Studnička, *Bericht über die mathematischen und naturwiss. Publicationen d. kön. Ger. d. Wissenschaften*, Prague, 1885, p. 11; *Vorlesungen über Geschichte der Mathematik*, vol. 4, ed. by M. Cantor, Leipzig, 1908; and Poggendorff, *Biographisch-literarisches Handwörterbuch*, vol. 2.

[3] *Philosophiae naturalis principia mathematica auctore Isaaco Newtono illustrata commentationibus . . .*, Prague, 1780 and 1785.

[4] Compare "Stanislav Vydra," *Časopis pro pěst. matem.*, vol. 1, 1872, pp. 1–6, 49–54; A. Rybicka, "Stanislav Vydra," *Časopis pro pěst. matem.*, vol. 44, pp. 53–68, 103–119; *Vorlesungen über Geschichte der Mathematik*, vol. 4, ed. by Cantor, Leipzig, 1908; Poggendorff, *Biographisch-literarisches Handwörterbuch*, vol. 2; *Oesterreichsche National-Encyylopädie*, vol. 6, Vinena, 1837; and F. M. Pelzel, *Böhmische, mährische und schlesische Gelehrte und Schriftsteller aus dem Orden der Jesuiten*, Prague, 1786.

[5] This, together with Jan Smolík, *Mathematikové v Čechách*, is the principal source of material for this article in respect to early mathematics—Q. VETTER.

devoting themselves to original research. For this end it was well there were several distinguished foreigners as professors at the University and Polytechnic Institute of Prague. We may name the Milanese Bernard Bolzano,[1] also Professor Wilhelm Fiedler,[2] Carl Küpper[3] and Jakob E. C. Durège.[4] A fierce struggle had ever to be maintained against the unjust Germanization of the school system. Individual cities founded Czech secondary schools, without the help of the government and even against the government. Only after long political controversy did they succeed in dividing the Polytechnic in 1869 and the University in 1882 into two institutions, the Czech and the German, and only at the end of the last century was a second Czech Polytechnic founded at Brno. Not until today has the Austrian yoke been thrown off, and now it is possible to erect as many Czech schools as the nation needs. It is natural that with such conditions many Czech scientists of the older generation, educated in German schools, wrote in German in order to make themselves known in the international scientific world.

The Czech mathematicians applied themselves with all their might to the difficult problem concerning which I have already written. In the second half of the last century several excellent Czech mathematicians flourished. In the history of mathematics Jan Smolík deserves notice. It is a pity that he forsook this field to devote himself to numismatics, and of his book *Matematikové v Cechách* (Mathematicians of Bohemia) in 1863, he published only the first part which concludes with the beginnings of the eighteenth century.

In foreign countries perhaps the best known are Emil and Eduard Weyr, two brothers. Emil, the older (1848–1894), having taken a journey through Italy for the purpose of study, met almost all the best mathematicians of the time; especially noteworthy was his friendship with L. Cremona, with whom he studied at Milan. On his return home he was made professor at the Polytechnic Institute of Prague and later at the University of Vienna.[5] Emil Weyr wrote many

[1] I have not been able to verify that there was any "Milanese" mathematician of this name. Of course the catholic theologian and philosopher Bernhard Bolzano (1781–1848) was born, and died, at Prague and was professor of the philosophy of religion in the University there for many years. He is the author of a number of well-known mathematical writings.—After this note was in type Dr. Vetter informed me that he wished the clause to read: "We may name Bernard Bolzano of an Italian family" . . . ; Bolzano's father was a native of Northern Italy.

[2] Otto Wilhelm Fiedler, born in Saxony in 1832, was professor of geometry at the Polytechnic Institute of Prague. He left there for a similar position in the Polytechnikum at Zürich where he remained till his death in 1912. He is chiefly known to Americans as the genial translator and editor of the German editions of George Salmon's works, and as the author of *Cyklographie oder Construction der Aufgaben über Kreise und Kugeln*, Leipzig, 1882.

[3] Carl Joseph Küpper (1828–1900), professor of geometry at the Polytechnic Institute, Prague. An obituary notice (accompanied by a portrait) with a list of his writings, by E. Waelsch, appeared in *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 14, 1905, pp. 389–394. See also Poggendorff, *Biographisch-literarisches Handwörterbuch*, vol. 4.

[4] Apparently instead of the initial "E." should be H. since Jacob Heinrich Karl Durège (1821–1893) was professor in the Polytechnic Institute, Prague, before 1869 when he went to Vienna. He lived in the United States 1851–57 and his German work on the theory of functions of a complex variable has been made familiar through the English translation (Philadelphia, 1896) of the late G. E. Fisher (see this MONTHLY, *1920*, 239) and I. J. Schwatt.

[5] Emil Weyr was born in Prague and was appointed professor of mathematics in the Univer-

important books on geometry, not only in Czech and in German, but also in French and Italian; his works brought him international fame. Projective geometry is indebted to him for many important discoveries. To Italians it may be interesting to know that he translated into Czech the memoirs of L. Cremona.[1] Eduard Weyr (1852–1903)[2] was also professor at the Czech Polytechnic Institute of Prague and not only enriched Czech mathematical literature but published articles in Italian, French, and German periodicals and made a name for himself which was respected in foreign scientific circles. Analytical geometry and algebra were the objects of his study.

Both brothers were active in the "Jednota českých mathematiků a fysiků" (Union of Czech mathematicians and physicists), the center of Czech mathematical life; during half a century this union has published a journal, *Časopis pro pěstování matematiky a fisiky* (Journal for the study of mathematics and physics), and several scientific manuals and text-books for secondary schools.

The most prolific writer among Czech mathematicians was František Josef Studnička (1836–1903),[3] the first professor of mathematics in the Czech University of Prague, where he taught for twenty years. A group of books and many memoirs came from his pen. His interest in science was very great; he wrote several books on meteorology. He was greatly interested in Czech terminology and tried to fill the gaps in Czech scientific literature. For that purpose he compiled the first Czech manuals of the different mathematical theories. To him belongs the credit for advancing history of mathematics at this time; he not only wrote several historical articles and published the works of the early mathematicians but also awakened among his pupils an interest in history. Teaching in secondary schools also attracted his attention, as we see in his memoirs and text-books. His greatest work however was that done in research in determinants.[4] Descriptive geometry was studied in Bohemia[5] with success. Of the

sity of Vienna in 1875. For material regarding his life and work the following sources may be consulted: *Časopis pro pěst. matem.*, vol. 24, 1895, pp. 163–244, by A. Pánek; *Monatshefte für Mathematik und Physik*, Vienna, vol. 6, 1895, pp. 1–4, by G. Kohn (Italian translation in *Rendiconti del Circolo Matematico di Palermo*, vol. 9, pp. 260–262); *Jahresbericht der deutschen Mathematiker-Vereinigung*, vol. 4, 1897, pp. 24–33, by G. Kohn; F. J. Obenrauch, *Geschichte der darstellenden . . . Geometrie*, Brno, 1897, pp. 365, 435 f.; *Živa. Sbornik vědecký musea kralovství Českého*, Prague, 1894, 14 pages, by A. Sucharda; and Poggendorff, *Biographisch-literarisches Handwörterbuch*, vols. 3 and 4.

[1] By "memoirs" is meant, possibly, the Czech translation of "Introduzione ad una teoria geometria d. curve piane" (Bologna, 1862); and of "Sulle transformazione geometriche delle, figure piane" (Bologna, 1862–1865), the Czech translation of the latter appearing it 1872.

[2] A biography and chronological list of writings of Eduard Weyr, by K. Petr and J. Sobotka is given in *Časopis pro pěst. matem.*, vol. 34, 1905, pp. 457–516. See also Poggendorff's *Biographisch-literarisches Handwörterbuch*, vols. 3 and 4.

[3] A sketch of his life and work, by A. Pánek, is given in *Časopis pro pěst. matem.*, vol. 33, 1904, pp. 369–480; see also vol. 32, 1903, p. 297. A fairly complete list of his writings is given in Poggendorff, *Biographisch-literarisches Handwörterbuch*, vols. 3 and 4.

[4] For an account of this, see T. Muir, *The Theory of Determinants in the Historical Order of Development*, vol. 3, 1861–1880, London, 1920, various references in the index.

[5] Concerning descriptive geometry and its literature in Bohemia up to 1883 one may consult two works by Vaclav Lavička: (1) *Historie deskriptivní geometrie*, Prague, 1878; (2) *Deskriptiva geometrie ze stanoviště historickopaedagogického, Sešit prvý* (Descriptive geometry from the historical pedagogical standpoint, part 1), Pardubitz, 1883.

Czech professors of this subject at the Polytechnic Institute of Prague,[1] the earliest is Rudolf Skuherský (1821–1863).[2] After the division of the Institute, in the Czech section František Tilšer (1825–1913) [3] was the first professor of descriptive geometry. He worked on the theory of illumination and he is author of a special theory, the theory of "ikonognosie," [4] in which he tries to connect descriptive geometry with philosophy. Karel Pelz (1845–1908) [5] is rather well known in foreign mathematical circles. His works excel in clever solutions of construction problems. The most celebrated of his works are those on axonometric projection.

Of other mathematicians now dead, I shall cite only the professors of both the Czech polytechnic institutes at Prague and Brno, Gabriel Blážek,[6] Augustin Pánek (1843–1908),[7] Karel Zahradnik (1848–1916),[8] who translated into Czech from Italian the works of Bellavitis,[9] Václav Řehořovský (1849–1911),[10] Anton Sucharda (1854–1906)[11] and František Velísek,[12] (1877–1914), who was killed in the Great War.

[1] For biographies of professors in this Institute the best work is A. V. Velflík, *Dějiny technikého učeni v Praze* (History of the Polytechnic Institute in Prague), vol. 1, 1906.—Q. VETTER.

[2] R. Shuherský was born in 1828, not 1821. He was professor at the Institute 1854–1863. For sketches of his life and work see F. J. Obenrauch, *Geschichte der darstellenden und projectiven Geometrie*, Brno, 1897, p. 353 f.; G. Loria, *Storia della Geometria Descrittiva dalle origini sino ai giorni nostri*, Milan, 1921, pp. 330, 340–341; and *Das ständisch-polytechnische Institut zu Prag. Programm zur fünfzigjährigen Erinnerungs-feier*. Ed. by C. Jelinek, Prag, 1856, pp. 250–251. Compare also Poggendorff, *Biographisch-literarisches Handwörterbuch*, vols. 2–3.

[3] A sketch of his life and work is given in G. Loria, *Storia della Geometria Descrittiva*, Milan, 1921, pp. 335–339, etc., and by B. Procházka in *Časopis pro pěst. matem.*, vol. 48, 1914, pp. 1–25. He was professor of geometry and astronomy at the Institute, 1864–1895. See also F. J. Obenrauch, *Geschichte der darstellenden . . . Geometrie*, Brno, 1897, p. 353 f., and Poggendorff, *Biographisch-literarisches Handwörterbuch*, vol. 3.

[4] "Grundlagen der Ikonognosie. I. Abteilung" by F. Tilšer, *Abhandlungen der königl. böhmischen Gesellschaft*, 1878, pp. 1–88.

[5] His work is discussed rather fully in G. Loria, *Storia della Geometria Descrittiva*, Milan, 1921, pp. 371–379, etc.; by J. Sobotka in *Časopis pro pěstov. matem.*, vol. 39, pp. 433–460; F. J. Obenrauch, *Geschichte der darstellenden . . . Geometrie*, Brno, 1897, p. 355 f. See also Poggendorff, *Biographisch-literarisches Handwörterbuch*, vols. 3–4.

[6] Born 1842; appointed ordinary professor in the Bohemian Polytechnic Institute, Prague, 1872.

[7] Professor of higher mathematics at the Bohemian Polytechnic Institute 1896–1908. Editor, 1884–1908, of *Časopis pro pěst. matem.*, also of the mathematical part of the Bohemian encyclopædia *Ottův Slovnik Naučný*. For a sketch by K. Petr, with a portrait, see *Časopis pro pěst. matem.*, vol. 41, 1912, pp. 1–8.

[8] Karel Dragutin Zahradnik, a prolific writer of German and Bohemian mathematical papers, especially in the field of geometry. A critical biography with a list of his writings, by J. Vojtěch, appeared in *Časopis pro pěstov. matem.*, vol. 46, 1917, pp. 235–303.

[9] That is, G. Bellavitis, *Methoda equipollencé čili rovnic geometrických*, Prague, 1874. In the same year was published a French translation by C. A. Laisant.

[10] An obituary notice of V. K. Rehořovský, by J. Sobotka, appeared in *Časopis pro pěst. matem.*, vol. 42, 1913, pp. 129–145.

[11] An obituary notice by K. Rychlík appeared in *Časopis pro pěst. matem.*, vol. 51, 1922, pp. 247–248.

[12] Professor of mathematics in the Polytechnic Institute, Brno, 1900–1907; he died February 19, 1907, according to *Bibliotheca Mathematica*, series 3, vol. 7, p. 424. He was the constructor of various types of models of cubic surfaces (see W. Dyck, *Katalog mathematischer und mathematisch-physikalischer Modelle, Apparate und Instrumente*, Munich, 1892, pp. 299–300.) A biography by J. Sobotka appeared in *Časopis pro pěst. matem.*, vol. 37, 1908, pp. 353–359. A number of papers by Sucharda are listed in Poggendorff's *Biographisch-literarisches Handwörterbuch*, vols. 3–4. See also G. Loria, *Storia delle Geometria Descrittiva*, Milan, 1921, pp. 357, 534, 547.

Today we have a group of mathematicians in the Czech universities of Prague and Brno and in the polytechnic institutes of these cities. Professor Lerch [1] is known for his works on mathematical analysis. Václav Láska,[2] originally a geodesist and professor of applied mathematics and physics, is interested also in geophysics, and in the philosophy, history and teaching of mathematics. Professor Jan Sobotka[3] is a geometrician known also beyond the limits of Bohemia. Professor Karel Petr[4] has given his attention to analysis and arithmetical questions. Geometry is the specialty of Professors Bohumil Bydžovský, Jan Vojtěch, Ladislav Seifert, and Bohuslav Hostinsky who now is devoting himself to applied mathematics and theoretical physics. Professor Karel Rychlík gives his time to algebra, Professor Josef Benés and Doctor Emil Schoenbaum to actuarial mathematics. The theory of functions is the special branch of Dozent Miloš Kössler. At the Polytechnic Institute of Prague are also Professors Matthias Vaněček [5] and Josef Klobouček, and Dozenten František Rádl and Karel Dusl. Professors Čenek Jarolímek [6] and Bedřich Procházka[7] have written several books

[1] Matyáš Lerch, born in Bohemia in 1860, professor of mathematics at the University of Freiburg, Switzerland, from 1896 till his appointment as professor in the Polytechnic Institute at Brno in 1906. Biographical notes may be found in *Annuario Biografico del Circolo Matematico di Palermo*, 1914, and in *Acta Mathematica 1882–1912, Table Générale des tomes 1–35*, Upsala, 1912, where there is also a portrait. For long lists of his writings see Royal Society of London, *Catalogue of Scientific Papers*, vol. 16; *Revue Semestrielle des Publications Mathématiques;* and Poggendorff's *Biographisch-literarisches Handwörterbuch*, vol. 4. The Academy of Sciences of the Institute of France awarded him in 1900 the grand prize for mathematical sciences. He died August 3, 1922.

[2] That is Václav Jan (=Wenzel Johann) Láska, professor of higher geodesy and astronomy, and director of the astronomical, meteorological and seismological observatory at the University of Lemberg, 1899–1911. He is best known to American mathematicians by his *Sammlung von Formeln der reinen und angewandten Mathematik*, Braunschweig, 1889–1894, 1090 pages. His other books and numerous papers written in German and Bohemian and published before 1904, and dealing with topics in mathematics, theoretical astronomy and mathematical geography, surveying, and geodesy, are listed in "Poggendorff," in Royal Society of London, *Catalogue of Scientific Papers*, vol. 16, and in *Revue Semestrielle des Publications Mathématiques*.

[3] Jan Sobotka was born in Bohemia in 1862. He was professor of descriptive geometry in the Technical Institute of Brno from 1899 till 1904 when he was appointed professor in the Bohemian University of Prague. See G. Loria, *Storia della Geometria Descrittiva*, Milan, 1921, various references in index; *Annuario Biografico del Circolo Matematico di Palermo*, 1914; Poggendorff's *Biographisch-literarisches Handwörterbuch*, vol. 4, and *Revue Semestrielle des Publications Mathématiques*. His *Deskriptivní Geometrie promítání parallelního*, vol. 1, was published in 1906 by the Society of Czech mathematicians as vol. 10 of its *Sbornik*. A biography and list of his writings, by F. Kadeřávek, appeared in *Časopis pro pěst. atem.* vol. 52, 1922, pp. 1–9 + portrait.

[4] K. Petr, professor of mathematics at the Bohemian University of Prague since 1903, was born in Bohemia in 1868. For a list of his papers see Royal Society of London, *Catalogue of Scientific Papers*, vol. 17; Poggendorff's *Biographisch-literarisches Handwörterbuch*, vol. 4; and *Revue Semestrielle des Publications Mathématiques*.

[5] Born 1859.

[6] Author of articles listed in Royal Society of London, *Catalogue of Scientific Papers*, vol. 16, and in *Revue Semestrielle des Publications Mathématiques*. His treatise on descriptive geometry was a three-volume work first published in 1875–1877 by the Jednota českých mathematiků. During the next thirty years it was almost the only text used in Bohemian Real Schools. It was translated into Bulgarian, by Šourek, for use in secondary schools. See G. Loria, *Storia della Geometria Descrittiva*, Milan, 1921. The Christian name "Čenek" is the same as "Vincenc." Since the above was in type Dr. Vetter furnished me with the following information: Professor Jarolímek was born in 1846 and died in 1921. He was professor at the Bohemian Polytechnic

on descriptive geometry. Professors Miloslav Pelíšek[1] and František Kadeřávek (b. 1885), and Dozent Josef Kaunovsky (b. 1881) teach this branch also. At the University of Prague, philosophy of mathematics is taught by Professor Karel Vorovka[2] (b. 1879) and the history of mathematics by the author of this article (b. 1881).

SOME CURIOUS FALLACIES IN THE STUDY OF PROBABILITY.

By ROBERT E. MORITZ, University of Washington.

Part II.

The Petersburg paradox[3] discussed in Part I consists in the conclusion that under the following conditions Paul's expectation is, by elementary rules, infinite. A coin is tossed until head turns up. If this happens on the first trial Peter is to pay Paul 1 crown, if on the second trial 2 crowns, if on the third 4 crowns, and so on, the sum Peter is to pay Paul when head finally does turn up being double the sum he should have paid had head turned up on the immediately preceding trial.

It is not necessary to recount the earlier attempts that were made to explain the Petersburg paradox. This has been done by the historians of the science of probability, by Todhunter in his *History of the Theory of Probability* from Pascal to Laplace by Czuber in *Grunert's Archiv*, volume 67, and by Netto in the fourth volume of Cantor's *Geschichte der Mathematik*. To the modern reader many of these attempts seem puerile if not ridiculous, as for instance Cramer's attempt to explain the paradox by considering all sums greater than 2^{24} as practically equal, or D'Alembert's distinction between physical and metaphysical possibilities and his contention that some such expression as $1/[2^n(1 + \beta n^2)]$ or $1/(2^n + 2^{an})$ rather than $1/2^n$ must be accepted as the probability that head turns up for the first time on the nth trial, or Bequelin's conclusion that the oftener a chance event has happened the less likely it is to happen on the next trial, or Buffon's doctrine that any probability less than 10^{-4} must be considered absolutely zero.

It is, however, not without interest to consider some of the explanations offered by more recent writers.

Institute, 1907–1915. A biography by J. Sobotka, with two lists of writings, is given in *Časopis pro pěst. matem.*, vol. 45, pp. 439–449, and vol. 51, 1922, pp. 67–68.

[1] Author of articles listed in Royal Society of London, *Catalogue of Scientific Papers*, vol. 7, and in *Revue Semestrielle des Publications Mathématiques*. M. Pelíšek was born in 1855; see Poggendorff's *Biographisch-literarisches Handwörterbuch*, vol. 4. Professor Procházka's Christian name "Bedřich" is the same as "Friedrich."

[2] Heft 13 (89 pages, Vienna, 1914) of the report on mathematical instruction in Austria to the International Commission on Mathematical Instruction was by K. Vorovka, L. Červenka and V. Posejpal, with a preface by J. Sobotka. The report is entitled *Die Lehrbücher für Mathematik, darstellende Geometrie und Physik an den Mittelschulen mit böhmischer Unterrichtssprache*, and contains considerable historical material. Other reports of this series give an appreciable amount of information concerning mathematics in Bohemian institutions; for example: *Der Unterricht in der darstellenden Geometrie an den Technischen Hochschulen* by E. Müller (Heft 9); *Der mathematische Unterricht an den Universitäten* by R. v. Sterneck (Heft 7); *Der mathematische Unterricht an den Technischen Hochschulen* by E. Czuber (Heft 5). For other publications of Professor Vorovka see *Revue Semestrielle des Publications Mathématiques*.

[3] For numerous references, see E. Czuber's article *Wahrscheinlichkeitsrechnung* in *Encyclopädie der Mathematischen Wissenschaften*, I D 1, Section 17. Translated and amplified by J. Le Roux in *Encyclopédie des Sciences Mathématiques*, I 20.

Bachelier[1] discusses the paradox but makes no effort to explain it. "Common sense," he says, "can not be invoked to settle delicate mathematical questions. Common sense is unable to determine whether the area between a curve and its asymptote is finite or infinite, or whether a series is convergent or divergent. In the Petersburg problem the stake which Paul is to receive if he wins at any one trial is double the stake he would have received had he won at the preceding trial; if the multiplier had been 1.999 instead of 2 his expectation would have been finite, for the series

$$\frac{1}{2}+\frac{1.999}{2^2}+\frac{(1.999)^2}{2^3}+\frac{(1.999)^3}{2^4}+\cdots$$

is convergent. Common sense, however, is unable to distinguish the difference."

This of course does not explain why there is such a discrepancy between common sense and the results of logical reasoning. Nor is the illustration very fortunate, for the sum of the above series is 1000, and common sense would forbid Paul to venture 1000 crowns, or any other considerable sum, for what he might reasonably expect to win on accepting the conditions of the play. The paradox consists not in that the expectation is infinite, but in that it is larger than common sense can admit it to be.

Czuber[2] takes the view that the explanation of the paradox is to be sought in the conditions of the problem, which specify a play which may never end and which may require Peter to pay out a sum greater than any possible sum. These are mere words, he says, without a real meaning. If the problem is to have a real meaning it must be limited to a definite maximum number of trials, and this limitation must be such that Peter can actually meet all possible losses. When the problem is thus modified no objection can be raised against the resulting expectation of Paul.

Czuber then goes on to show that if the play were limited to n trials with the understanding that if head does not turn up during the first n trials Peter must give Paul 2^n crowns, since that is the least sum Paul could win if the play were continued indefinitely, Paul's expectation would be

$$\frac{1}{2}+\frac{2}{2^2}+\frac{2^2}{2^3}+\frac{2^3}{2^4}+\cdots+\frac{2^{n-1}}{2^n}+\frac{2^n}{2^n}=\frac{n}{2}+1.$$

Suppose that Peter had a million crowns, the maximum loss that he could pay would then be $2^{19} = 524{,}288$ crowns. It would therefore be necessary to limit the play to 19 tosses[3] and Paul's expectation would be only 10.5 crowns.

Now neither of the objections which Czuber raises, namely that the Petersburg problem is necessarily meaningless because it deals with a play that may never end, and that may require Peter to pay out a sum greater than he can possibly possess, objections which, by the way, were previously raised by both

[1] L. Bachelier, *Calcul des Probabilités*, vol. 1, Paris, 1912, p. 26.

[2] E. Czuber, *Wahrscheinlichkeitsrechnung*, Leipzig, 1903, p. 203; also in *Grunert's Archiv*, vol. 67.

[3] Czuber says 20, but that is evidently an error.

Poisson and Buffon, are valid objections. For nothing can prevent us from assuming that Peter has unlimited wealth, or that instead of playing for real money the players play for stage money, or counters, or marks of ink on paper to symbolize money. Nor does it follow that the expectation must be meaningless because of the fact that the play may never end, for there are many problems which impose this condition in which the resulting expectation is perfectly compatible with common sense. Let us assume that Peter is to pay Paul one crown as soon as head turns up for the first time. Then, clearly, this play also may never end, and Paul's expectation is

$$\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots=1$$

which is in perfect agreement with common sense. The conditions to which Czuber objects do not, therefore, necessarily imply a paradox.

De Morgan in his admirable *Essay on Probabilities* (London, 1838), like Condorcet before him, held that the Petersburg problem offered really no paradox and tried to show that the conclusion that Paul's expectation is infinite is consistent with the results of experience. He cited the results of Buffon's experiment which consisted of 2,048 sets of tosses of a coin. In his *Formal Logic* he gave the results of another 2,048 sets of trials made by one of his own pupils. In the combined number of 4,096 sets of trials head showed on the first toss in 2,109 sets, on the second in 1,001 sets, on the third toss in 480 sets, and so on. The complete results are exhibited in the following table.

	Results of Trials.		Theoretical Results.	
	Games Won.	Amount Won.	Games Won.	Amount Won.
On 1st toss	2,109	2,109	2,048	2,048
" 2d "	1,001	2,002	1,024	2,048
" 3d "	480	1,920	512	2,048
" 4th "	236	1,888	256	2,048
" 5th "	127	2,032	128	2,048
" 6th "	67	2,144	64	2,048
" 7th "	42	2,688	32	2,048
" 8th "	17	2,176	16	2,048
" 9th "	11	2,816	8	2,048
" 10th "	3	1,536	4	2,048
" 11th "	1	1,024	2	2,048
" 12th "	0	0,000	1	2,048
" 13th "	0	0,000		
" 14th "	1	8,192		
" 15th "	0	0,000		
" 16th "	1	32,768		
Total	4,096	63,295	4,096	24,576
Average per game		15.4		6

According to the theory, Paul in playing 4,096 games could have expected to win an average of 6 crowns per game; the actual trials gave him an average of 15.4 crowns per game. Buffon's 2,048 sets of trials would have given 5 crowns

per game as against the theoretical result of 5.5 crowns per game. De Morgan, in discussing the results of Buffon's trials, calls attention to the significant fact that the rarer occurrences contributed most to the total amount. "If Buffon had tried a thousand times as many games, the results would not only have given more, *but more per game*. The larger net would have caught, not only more fish, but more varieties of fish; in two million of sets, we might have expected to have seen cases in which head did not appear till the twentieth throw. . . . Thus the reader may readily perceive that the player might continue until he had realized not only any given sum, but any given sum *per game;* a result which is indicated by the application of our rule, when it tells us that the mathematical expectation of the player upon a single game is infinite."

Theoretically, as is obvious by inspecting the two right-hand columns in the foregoing exhibit, 2^n games should yield on an average $n/2$ crowns per game. To secure an average of 18 crowns per game would therefore require 2^{36} games, a number of games exceeding the number of seconds in the Christian era. Theoretically the games in which head turns up for the first time on the twelfth toss contribute exactly the same sum toward the total amount as the games in which head turns up at the first trial. De Morgan, therefore, seems to have failed to grasp the real significance of his own observation to the effect that the rarer occurrences contributed more to the total amount won than did the more frequent occurrences, and overlooked completely the even more significant fact that, contrary to the general rule, the average (15.6) attained from 4,096 sets of trials deviated more from the theoretical average (6) than did the average (5) resulting from Buffon's 2,048 sets from the corresponding theoretical average (5.5).[1] This fact points to a greater and greater deviation from the theoretical average as we increase the number of trials. If this should be the case, the mathematical expectation, as computed by the general formula derived for a finite number of contingent prospects, would be meaningless when applied to the Petersburg problem.

It is a cause of no little surprise that this observation has escaped the attention of all commentators on this subject. The mathematical expectation of a contingent prospect is by definition a mean value. It is not the value of any single hazard but the average value of a sufficiently large number of equal hazards. The mathematical expectation of one chance out of a thousand to secure a billion dollars is one million dollars, but this does not mean that anyone in his senses would pay a million dollars for a single chance of winning the billion dollars; a million dollars is the equivalent of such a chance only if the risk can be repeated sufficiently often to secure an average. If on repeating a risk sufficiently often, or it may be indefinitely, an average can not be secured, the formula for the mathematical expectation loses its significance. Now this is exactly what happens in the Petersburg problem. Let us suppose that n is a number so large that

[1] In his *Budget of Paradoxes* (Original Edition, p. 170) De Morgan cites the results of two other sets of 2,048 trials each. The combination of these results with those already cited, that is, of 8,192 trials in all, would yield an average of 14.6 crowns per game as against a theoretical average of 6.5 crowns per game.

n games will secure an average. Every game that is played involves the prospect of head not showing up till the nth toss of the coin, the probability of this prospect is $1/2^n$; this particular prospect will therefore be realized on an average but once in 2^n games. But this is contrary to the assumption that n games will secure an average. The foregoing reasoning is valid even if n is assumed equal to ∞, for each game involves events whose probability is $1/2^\infty$, and such events will occur on an average but once in 2^∞ games. The conjecture, occasioned by Buffon's and De Morgan's experimental results, that the formula for the mathematical expectation when applied to the Petersburg problem leads to a meaningless result is thus verified.

Whitworth[1] ignores the infinite result given by the formula for the mathematical expectation and proceeds to determine the unknown sum x which Paul might risk on the play on the condition that if the play were repeated each time on a scale proportional to his capital at that time, he would in the long run be left neither richer nor poorer. His procedure is in substance as follows:

Let x denote the sum which Paul with an initial capital of C may pay for a chance p of winning a prize P. If he wins, his capital will be $C + P - x$, if he loses $C - x$, so that the effect of winning is to multiply his capital by the factor $(C + P - x)/C$, that of losing to multiply it by $(C - x)/C$.

If Paul repeats the venture n times, n being large enough to secure an average, he will win pn times and lose qn times, where $p + q = 1$. On the condition that the sum he risks in any one venture is proportional to the capital he then has, the result of the n operations will be to multiply his initial capital by the factor

$$\left(\frac{C + P - x}{C}\right)^{pn}\left(\frac{C - x}{C}\right)^{qn}.$$

Since now the result of the n operations is to leave Paul neither richer nor poorer, this factor must equal unity, hence also

$$\left(\frac{C + P - x}{C}\right)^{p}\left(\frac{C - x}{C}\right)^{q} = 1.$$

This equation immediately suggests the corresponding equation in which x denotes the sum which Paul may pay for the chance of winning one of k prizes $P_1, P_2, \cdots, P_k$, with the respective probabilities $p_1, p_2, \cdots, p_k$ of winning them, where $p_1 + p_2 + \cdots + p_k = 1$. The condition that in the long run Paul is neither richer nor poorer becomes

$$\left(\frac{C + P_1 - x}{C}\right)^{p_1}\left(\frac{C + P_2 - x}{C}\right)^{p_2}\cdots\left(\frac{C + P_k - x}{C}\right)^{p_k} = 1.$$

This equation does not admit of a general solution, but if x *is small as compared with* C it may be approximated to any desired degree of accuracy. As

[1] W. A. Whitworth, 5th edition, Cambridge, 1901, p. 243.

the result of such approximation, Whitworth gives without proof

$$x = \frac{\left(1+\frac{P_1}{C}\right)^{p_1}\left(1+\frac{P_2}{C}\right)^{p_2}\left(1+\frac{P_3}{C}\right)^{p_3}\cdots - 1}{\frac{p_1}{C+P_1}+\frac{p_2}{C+P_2}+\frac{p_3}{C+P_3}+\cdots}.$$ [1]

In the Petersburg problem the prizes are 1, 2, 2^2, 2^3, $\cdots$, and the respective chances of winning them are $(1/2)$, $(1/2)^2$, $(1/2)^3$, $(1/2)^4$, $\cdots$, so that in this case the equation for determining x becomes

$$\left(\frac{C+1-x}{C}\right)^{1/2}\left(\frac{C+2-x}{C}\right)^{1/4}\left(\frac{C+4-x}{C}\right)^{1/8}\cdots = 1,$$

and the approximate solution

$$x = \frac{\left(1+\frac{1}{C}\right)^{1/2}\left(1+\frac{2}{C}\right)^{1/4}\left(1+\frac{4}{C}\right)^{1/8}\cdots - 1}{\frac{1}{2(C+1)}+\frac{1}{4(C+2)}+\frac{1}{8(C+4)}\cdots}.$$

From this formula Whitworth finds that when $C = 8$, $x =$ about 3.8; when $C = 32$, $x =$ about 4; and when $C = 1{,}024$, $x =$ about 6, results which are in remarkable close agreement with those obtained from Bernoulli's hypothesis of moral expectation.[2]

However much Whitworth's highly ingenious solution may compel our admiration, it can not be considered a satisfactory solution of the Petersburg problem for at least three reasons.

First, it introduces a new condition into the problem, namely, that the price which Paul may pay for the advantage which Peter offers him is to be such that if the play were repeated each time on a scale proportionate to his funds at that time, he would in the long run come out even. Now it can be readily shown that any player who repeats a play, which is otherwise fair, on a scale proportionate to his capital, is sure to lose in the long run, therefore to stipulate that the x which Paul is to pay for his advantage is to be such as to leave him neither richer nor poorer if he repeats the play in this manner, is equivalent to asking odds in favor of Paul.

Suppose a player tosses a coin staking each time one half of what he possesses on the chance of heads turning up. If he wins on the first toss, he will have increased his funds by one half; if he tosses again and loses, he will have diminished

[1] This seems to be incorrect. The first approximation to x is

$$\frac{\log_e\left[\left(1+\frac{P_1}{C}\right)^{p_1}\left(1+\frac{P_2}{C}\right)^{p_2}\cdots\right]}{\frac{p_1}{C+P_1}+\frac{p_2}{C+P_2}+\cdots}.$$

[2] The correct approximation formula given in the preceding footnote yields the values $x = 2.85, 3.45, 6.28$ for $C = 8, 32, 1{,}024$ respectively.

his funds by one half, so that he will have left three fourths of what he originally possessed. If he loses the first toss, he will have left one half of what he had; if he wins the second toss, he will increase this half by one half of itself, so that again he will have left only three fourths of what he originally possessed. Suppose now that he continues to play until he has secured an average, that is, until he has won as often as he has lost, say $2n$ games in all. He will then have lost n games and won n games. Each game he loses will diminish his fortune by one half, each game he wins will increase it by one half. The result is that in the end he will have his original fortune multiplied by the factor $(1/2)^n(3/2)^n$, that is by $(3/4)^n$. If $n = 10$, $(3/4)^n$ is less than one sixteenth, so that, although the game is fair in every other respect and the number of games sufficiently large to secure an average, the player will have lost more than 15/16 of his original fortune.

It is commonly supposed that a player must necessarily win in the long run provided the odds are in his favor. This itself is a fallacy, as the following illustration will show. Suppose that a player plays for even stakes, always staking one half of his funds on a hazard in which the odds are 3 to 2 in his favor, that is, such that, in the long run, he will win 3 times out of 5. Suppose that he plays $5n$ games, n being sufficiently large to secure an average. He will then have won $3n$ games and lost $2n$ games. Each game he wins increases his fund by one half, each game he loses diminishes his fortune by one half. If his initial fund be denoted by C, his fund after playing $5n$ games will be

$$(1/2)^{2n}(3/2)^{3n} \cdot C = (27/32)^n \cdot C$$

which is certainly less than C. If $n = 10$, $(27/32)^n$ is less than 1/5, so that though the odds are 3 to 2 in favor of the player, he will have lost more than 4/5 of his original fortune.

In the second place it is obvious that the condition which Whitworth imposes may be replaced by any one of an infinite number of similar conditions and we should thus be led to an infinite number of different solutions of the problem. Instead of repeating the play on a scale proportionate to his funds at the time, the price Paul should pay might be determined on the condition that, if he repeated the play on some increasing or decreasing scale, he should come out even in the long run, the only essential consideration being that the scale be such that he may continue the play indefinitely without exhausting his fund.

Third, granting that the value of x fairly represents Paul's advantage, it remains to find x. Whitworth approximates the value of x on the assumption that x is small as compared with C, that is, so small that the second and higher powers of the quantity x/C, which enter into the expansion of the infinite product, may be neglected. This begs the question. In other words, Whitworth finds that x is small as compared with C on the assumption that x is small as compared with C.

The view which has been advanced in this paper, that the rule for the mathe-

matical expectation gives a meaningless result when applied to the Petersburg problem, explains the paradox but does not solve the problem. Indeed it may be safely asserted that the problem does not admit of solution without the aid of assumptions not warranted by the expressed conditions of the problem.

THE ORIGIN OF THE SYMBOLS FOR "DEGREES, MINUTES AND SECONDS."

By FLORIAN CAJORI, University of California.

At the present time there is a conflict of views regarding the origin of the symbol ° for "degrees" in angular measure. Edward O. von Lippmann[1] gives M as the symbol for the Greek word μοῖραι standing for parts, including parts of circular periphery, and then adds, "woraus wohl das Zeichen ° für Kreisgrad entstand" (from which doubtless originated the sign ° for circular degree). On the other hand, J. Tropfke[2] assigns the origin of °, not to Greek antiquity, but to the sixteenth century, and connects it, not with Greek symbolism, but with the exponential concept as applied to different units in the sexagesimal system. Tropfke cites L. Schoener (1586) as the earliest author he could find who used exponents for geometric series, thus:

III	II	I	0	I	II	III
216000.	3600.	60.	1	$\frac{1}{60}$	$\frac{1}{3600}$	$\frac{1}{216000}$

and for sexagesimal integers and fractions, thus:

"IIae	Iae	0	I	II	III"
3.	15.	7.	50.	34.	23.

As between Lippmann and Tropfke, the researches I have made decidedly favor the general conclusions of Tropfke, but I have found earlier dates than his for the first appearance of the signs ° ′ ″.

In Athelard of Bath's translation into Latin (twelfth century) of certain Arabic astronomical tables, the names signa, gradus, minutæ, secundæ, etc., are abbreviated. The contractions are not always the same, but the more common ones are "Sig.," "Gr.," "Min.," "Sec."[3] No symbolism is used here in the designation of angular measure. Neither here nor elsewhere have I found indications of notations in support of Lippmann's explanation of the origin of the sign °. In the Alfonsion Tables[4] one finds marked by $49^{\ddot{2}}\ 32^{\ddot{1}}\ 15^{\ddot{m}}\ 4^{\ddot{2}}$ the sexagesimals $49 \times 60 + 32 + 15 \times \frac{1}{60} + 4 \times \frac{1}{60^2}$. It was during the six-

[1] E. O. von Lippmann, *Entstehung und Ausbreitung der Alchemie*, Berlin, 1919, p. 353.

[2] Johannes Tropfke, *Geschichte der Elementar-Mathematik*, 2. ed., Leipzig, vol. 1, 1921, p. 43.

[3] See H. Suter, *Die Astronomischen Tafeln des Muḥammed ibn Mūsā Al-Khwārizmī in der Bearbeitung des Maslama ibn Aḥmed Al-Madjriti und der latein. Uebersetzung des Athelard von Bath.* Copenhagen, 1914, p. xxiv.

[4] A. Wegener in *Bibliotheca mathematica*, series 3, vol. 6, 1905, p. 179.

teenth century that the modern notation for degrees, minutes and seconds took form.

Oronce Fine[1] in 1535 used for them the marks *grad*, $\bar{m}$, $\bar{2}$, $\bar{3}$, ; Gemma Frisius[2] in 1540 wrote

Integr.	Mi.	2.	3.	4.	
36.	30.	24	50	15	for our modern $36° 30' 24'' 50''' 15^{iv}$,

but in the revised 1569 edition of his book published in Paris there is an appendix on astronomical fractions, due to J. Peletier (dated 1558), where one finds,

"Integra, Mi vel $\overset{\circ}{1}$, $\bar{2}$, $\bar{3}$, $\bar{4}$, $\bar{5}$, $\bar{6}$, $\bar{7}$, $\bar{8}$, etc."
$\overset{\circ}{1}$, 2, 3, 4, 5, 6, 7, 8.

This is the first appearance that I have found of ° for integra or degrees. It is explained that the denomination of the product of two such denominate numbers is obtained by combining the denominations of the factors; minutes times seconds give thirds, because $1 + 2 = 3$. The denomination ° for integers or degrees is necessary to impart generality to this mode of procedure. "Integers when multiplied by seconds make seconds, when multiplied by thirds make thirds" (folio 62, 76). It is possible that Peletier is the originator of the ° for degrees. But nowhere in this book have I been able to find the modern angular notation ° ′ ″ used in writing angles. The ° is used only in multiplication. The angle of 12 minutes and 20 seconds is written "S.o.$\bar{g}$.0 $\bar{m}$.12.$\bar{2}$.20." (folio 76).

Twelve years later (1670) one finds in a book of Johann Caramuel[3] the signs ° ′ ″ ‴ ^{iv} used in designating angles. In 1571 Erasmus Reinhold[4] gave an elaborate explanation of sexagesimal fractions as applied to angular measure and wrote "°63 ′13 ″53," also "62° 54′ 18″." The positions of the ° ′ ″ are slightly different in the two examples. This notation was adopted by Tycho Brahe[5] who in his comments of 1573 on his *Nova Stella* writes 75° 5′, etc.

As pointed out by Tropfke, the notation ° ′ ″ was used with only minute variations by L. Schoener (1586), Paul Reesen (1587), Raymarus Ursus (1588), Barth. Pitiscus (1600), Herwart von Hohenburg (1610), Peter Crüger (1612), Albert Girard (1626). The present writer has found this notation also in Rhaeticus[6] (1596), Kepler[7] (1604), and W. Oughtred.[8] But it did not become universal. In later years many authors designated degrees by "Grad." or "Gr.," or "G."; minutes by "Min." or "M."; seconds by "Sec." or "S."

[1] Orontius Finaeus *Arithmetica practica*, Paris, 1535, p. 46. We quote from Tropfke, *op. cit.*, p. 43.

[2] Gemma Frisius, *Arithmeticae practicae methodus facilis*, Strasbourg, 1559, p. 57 v°. From Tropfke, *op. cit.*, p. 43.

[3] Joannis Caramvelis, *Mathesis Bicepts Vetus, et Nova*, Companiae, 1670, p. 61.

[4] E. Reinhold, *Prvtenicae tabvlae coelestivm motvvm*, Tübingen, 1571, folio 15.

[5] Tycho Brahe, *Opera omnia*, ed. I. L. E. Dryer, vol. 1, Copenhagen, 1913, p. 137.

[6] *Opus Palatinvm de triangvlis a Georgio Joachimo Rhetico coeptvm.* 1696, p. 3 ff.

[7] J. Kepler, *Ad vitellionem paralipomena quibus astronomiae pars optica traditur.* Frankfurt, 1604, p. 103, 139, 237.

[8] William Oughtred in *Clavis Mathematicae*, 1631; Anonymous Appendix to E. Wright's 1618 translation of John Napier's *Descriptio.*

QUESTIONS AND DISCUSSIONS.

EDITED BY C. F. GUMMER, Queen's University, Kingston, Ont., Canada.

DISCUSSIONS.

Not long ago Professor H. S. Uhler described in this MONTHLY (*1921*, 447) a plan for the multiplication of large numbers without undue expenditure of labor. The principal feature of the method was the preparation of nine strips of cardboard containing the products of the multiplicand by the digits 1, 2, ···, 9, from which these products might be obtained and inserted in their usual places, as often as they occurred, by mere copying. The saving of labor would be due to the fact that with a large multiplier the same partial products would appear a number of times. Since the appearance of Professor Uhler's note, several communications on the same subject have come in to this department.

It is evident that, if we were under the unfortunate necessity of carrying out multiplications involving thousands of figures, we might find it worth while to prepared in advance the products of the multiplicand by numbers up to 99, or higher. Actually, however, we may spare ourselves this task by use of a calculating machine. The notes printed below by D. N. Lehmer, J. P. Ballantine, and D'A. W. Thompson are concerned with the problem as modified by this consideration.

In the second discussion, Professor Bradley presents in a convenient form the solution of the celebrated problem on triangles with integral sides and area.

I*a*. ON THE MULTIPLICATION OF LARGE NUMBERS.

By D. N. LEHMER, University of California.

In certain researches in the theory of cubic irrationalities it has been necessary to compute determinants of the third order whose elements are of the order of twenty digits. Since I was interested in the factors of the result, it was necessary to obtain the product to the very last digit. I have found for this purpose a multiplying machine to be indispensable even when the factors are far larger than the capacity of the machine. By a slight modification of the method of cross-multiplication which is described in *Science*, volume 16, July, 1902, page 71, I was able to perform the multiplication given by Professor Uhler in this MONTHLY, *1921*, p. 447, in half an hour, using an eight place Monroe Calculating Machine. For such a machine I separate the numbers into periods of eight digits, and write one over the other thus:

23140692	63277926	90057290	86367948	54738026	61062426	00211600
43213918	26377224	97744177	37171728	01127572	81098106	33085400
9999999	66551256	78226922	68470128	58244096	21798502	61794556
	33448743	21773076	31529870	41755902	78201496	38205832
		1	1	1	1	1
10000000	00000000	00000000	00000000	00000000	00000000	00000389

The above is all the work that is put down in the actual computation. I obtain first the product

$$23140692 \times 43213918 = 9999999\ 66551256$$

which I set down in the third line. Then since the machine can compute $AB + CD$ without taking anything out of the machine, I obtain $23140692 \times 26377224 + 63277926 \times 43213918 = 33448743\ 21773076$. This result I put down in the fourth row, eight places to the right of the first product in the third line. I then obtain on the machine the sums $23140692 \times 97744177 + 63277926 \times 26377224 + 90057290 \times 43213918 = 78226922\ 68470128$ and set this result down eight places to the right of the preceding product sum. The machine then gives me, $23140692 \times 37171728 + 63277926 \times 97744177 + 90057290 \times 26377224 + 86367948 \times 43213918 = 1\ 31529870\ 41755902$. This I put down eight places to the right as before, the 1 being put in an additional row. The process is carried on in this way, eight new digits being added to the product at every step. The final addition is then made. The result differs a little at the end from Professor Uhler's, because he used digits beyond the fifty-two places which he has written down.

The method might be used with Crelle's *Rechentafeln*, only the numbers would be separated into periods of three, and the additions would have to be made "by hand." It is nevertheless a much better method for most purposes than the old methods.

I*b*. Note on the Multiplication of Long Decimals.

By J. P. Ballantine, Chicago, Ill.

We will suppose a calculating machine (such as the Monroe) which will carry 6, 8 or 10 digits on the keyboard, and twice as many on the carriage, denoting these numbers by μ and 2μ. We will denote by a any positive integer of μ or fewer digits, and will understand by a_1a_0 the number $a_1 \cdot 10^{\mu} + a_0$. Thus if a number contained 2μ digits at the left of the decimal point and 6μ at the right, it could be written in the form $a_1a_0 \cdot a_{-1}a_{-2}a_{-3}a_{-4}a_{-5}a_{-6} = \sum_{i=-6}^{1} a_i 10^{\mu i}$.

The multiplication of two numbers follows the ordinary laws of multiplication of two polynomials, thus:

$$A = a_0 \cdot a_{-1}a_{-2}a_{-3}, \qquad A' = a_0' \cdot a_{-1}'a_{-2}'a_{-3}',$$
$$A \times A' = (a_0 \times a_0') \cdot (a_0 \times a_{-1}' + a_{-1} \times a_0')(a_0 \times a_{-2}' + a_{-1} \times a_{-1}' + a_{-2} \times a_0') \cdots.$$

The combinations, of which the expression $(a_0 \times a_{-2}' + a_{-1} \times a_{-1}' + a_{-2} \times a_0')$ is typical, can be computed on the machine very readily. The products $a_0 \times a_{-2}'$, $a_{-1} \times a_{-1}'$, and $a_{-2} \times a_0'$ are taken, and if the carriage is not cleared between the operations, they are automatically added. The result will be a number of 2μ digits, and hence can be written in the form $a_2''a_1''$. The a_1'' is written down as part of the product $A \times A'$ (in the proper place), while the a_2''

is carried, that is, entered in the carriage of the machine to be taken as part of the next combination to the left.

In a precisely similar fashion it is possible to perform any of the familiar processes, such as long division, square root, and even Horner's method. It is as if the radix were 10^{α}, and instead of our memorizing the multiplication table up to $10^{\alpha} \times 10^{\alpha}$, the entries in this table were read from the machine.

To give a comparison between the longhand method and the machine method of multiplying, there are $\frac{1}{8}$ as many possibilities of error when an eight-place machine is used as when the work is done by hand; and, as for time, while Professor Uhler spent an afternoon on his multiplication, it takes by machine less than 20 minutes.

Ic. On the Multiplication of Large Numbers.

By D'Arcy W. Thompson, St. Andrews University.

The method of multiplying large numbers described by Professor H. S. Uhler is one which I happened to hit upon several years ago, and which I have used frequently. But it may be further simplified.

First, it is not necessary to *write* out the figures on coördinate paper, nor to transfer them to strips of cardboard. For the typewriter is itself an automatic "coördinator," and to typewrite the figures is at once to space them properly.

Secondly, if one performs the necessary summation by means of a Brunsviga, *no preliminary multiplication* is necessary; one does all that in the process of summation, by giving at each step the proper number of turns to the handle of the machine. So one has merely to type the multiplicand again and again, each time one space further to the right, and then place the multiplier vertically at the right-hand side and follow its indications as one turns the handle. Professor Uhler's example then is typed as follows:

```
4 3 2 1 3 9 1 8 2 · · · · · ×2
  4 3 2 1 3 9 1 8 · · · · · ×3
    4 3 2 1 3 9 1 · · · · · ×1
      4 3 2 1 3 9 · · · · · ×4
        · · · · · · · · · · · ·
```

Lastly, as a further improvement, the multiplier is typed (vertically) on a separate piece of paper, which serves as a movable rider, to be slid along the multiplicand as block after block of the latter is successively dealt with by the Brunsviga; and this rider has a slot which reveals just so broad a block of figures as may correspond to the capacity of the machine. Using this slotted rider, one does away with the necessity of blocking out the multiplicand by columnar interspacing, as in Professor Uhler's example. The rows of figures become thereby easier to write, and much easier to check, for the eye runs over identical figures in diagonal lines.

Professor Uhler says that his long multiplication of 50 digits by 50 was done in an afternoon and evening. With an assistant to dictate the numbers, I think it could be done very easily in an hour.

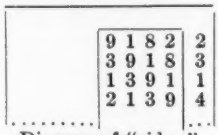


Diagram of "rider."

II. Rational Oblique Triangles.

By H. C. Bradley, Massachusetts Institute of Technology.

In this Monthly, *1921*, page 246, Professor Dickson gives Euler's results for the sides of a rational oblique triangle. Omitting the arbitrary constant multiplier, these are

$$x = \frac{m^2 + n^2}{mn}, \qquad y = \frac{M^2 + N^2}{MN}, \qquad z = \frac{(mN \pm nM)(mM \mp nN)}{mnMN},$$

where m and n are relatively prime integers, not both odd, and where M and N are relatively prime integers, not both odd.

These formulæ are not well adapted for finding integral solutions. However, let us multiply each by the factor mMn^2N^2. This gives

$$x = Mn(m^2N^2 + n^2N^2), \qquad y = mN(M^2n^2 + n^2N^2),$$
$$z = (mN \pm Mn)(mMnN \mp n^2N^2).$$

Now let $mN = p$, $Mn = q$, $nN = r$, and we have the simple three parameter solution

$$x = q(p^2 + r^2), \qquad y = p(q^2 + r^2), \qquad z = (p \pm q)(pq \mp r^2).$$

If from these three sides we find the area by the formula $A = \sqrt{s(s-x)(s-y)(s-z)}$, the result is

$$A = pqrz.$$

These results are the same as in Carmichael's *Diophantine Analysis* (New York, 1915), page 12, except that he does not give the double sign in the value of z. While this omission causes no loss in generality, it makes solutions in small integers less readily obtainable.

In using these formulæ, a few points may be noted. If we take $p = q$, only isosceles triangles are obtained. If we take either $p = r$, or $q = r$, only right triangles will result. Hence for general oblique triangles, all sides unequal, take $p \neq q \neq r$, though any two may have a common factor. Even then the equations occasionally give a right or an isosceles triangle. A few interesting cases are appended.

x	y	z	A
4	13	15	24
9	10	17	36
3	25	26	36
7	15	20	42
6	25	29	60
11	13	20	66
5	29	30	72
10	17	21	84
13	14	15	84
12	17	25	90

RECENT PUBLICATIONS.

REVIEWS.

The Cambridge Colloquium, 1916. Part 2: Analysis Situs. By OSWALD VEBLEN. (American Mathematical Society Colloquium Lectures, volume 5, part 2.) New York, 1922. 150 pp. Price $2.00 (unbound).

Extracts from Preface: "The Cambridge Colloquium lectures on Analysis Situs were intended as an introduction to the problem of discovering the n-dimensional manifolds and characterizing them by means of invariants. For the present publication the material of the lectures has been thoroughly revised and is presented in a more formal way. It thus constitutes something like a systematic treatise on the elements of Analysis Situs. The author does not, however, imagine that it is in any sense a definitive treatment. For the subject is still in such a state that the best welcome which can be offered to any comprehensive treatment is to wish it a speedy obsolescence.

"The definition of a manifold which has been used is that which proceeds from the consideration of a generalized polyhedron consisting of n-dimensional cells. The relations among the cells are described by means of matrices of integers and the properties of the manifolds are obtained by operations with the matrices. The most important of these matrices were introduced by H. Poincaré, to whom we owe most of our knowledge of n-dimensional manifolds for the cases in which $n > 2$. But it is also found convenient to employ certain more elementary matrices of incidence whose elements are reduced modulo 2, and from which the Poincaré matrices can be derived.

"The operations upon the matrices lead to combinatorial results which are independent of the particular way in which a manifold is divided into cells and therefore lead to theorems of Analysis Situs. . . .

"It will be seen that, aside from this one question which has to be dealt with in order to give significance to the combinatorial treatment, we leave out of consideration all the work that has been done on the point-set problems of Analysis Situs and on its foundation in terms of axioms or definitions other than those actually used in the text. We have also been obliged by lack of space to leave out all reference to the applications. . . ."

Contents—Chapter I: Linear graphs, 1–33; II: Two-dimensional complexes and manifolds, 34–72; III: Complexes and manifolds of N dimensions, 73–99; IV: Orientable manifolds, 100–124; V: The fundamental group and certain unsolved problems, 125–150.

No review, in the usual sense, of this work will be appropriate in this MONTHLY. Many of the readers of the MONTHLY will not be interested in a critical estimate of the importance, originality, or accuracy of this significant series of lectures. A few remarks to supplement the statements in the above given extracts from the "Author's Preface" will suffice here.

The study of Analysis Situs is the study of properties which remain invariant under continuous transformations. These may be roughly divided into "im Kleinen" and "im Grossen" properties, that is, properties which have reference to the infinitesimal character of the figure under consideration, and those which derive their significance from the figure as a whole. The distinction is a qualitative one which may be difficult to apply in certain cases, but in many familiar instances, from every mathematical study employing notions of continuity, this separation corresponds to an actual division of subject matter and of methods of attack. One might call these, respectively, the differential and the integral properties. In but one dimension, the major part of the theory of sets of points may be properly counted under Analysis Situs (of the differential sort), while the classification of different types of closed one-dimensional continua, with

respect to internal properties alone, is trivial in the extreme. The handling of sets of points in the plane, a workable definition of a "curve," considerations of equi-continuity, and so forth, present all kinds of unexpected problems and even paradoxes. Space-filling curves, curves of content, the proof that a simple closed curve divides the plane into two regions, and similar topics rightly suggest a host of delicate questions of rigor. All of these fall outside of the scope of these lectures.

The fact that there are actually questions of Analysis Situs in the integral sense, in two dimensions, is usually brought to vivid consciousness for the first time by the exhibition of a twisted paper strip of one continuous edge and one continuous surface. Since such "applications" are largely missing from the present work, and since the concrete problems are at once translated into equivalent problems in finite combinatorial analysis, the merely casual reader may find difficulty in seeing the connection between most of the topics here discussed and continuous transformations. The study is really one of matrices, the permissible operations being those suggested by the geometry of the problem, while little further mention is made of the geometrical interpretation of the several steps. By this procedure, dependence upon spatial preconceptions is largely eliminated and the subject acquires a degree of mathematical rigor, with flexibility of manipulation, that is a refreshing surprise to most mathematicians who approach this study for the first time.

Except for the applications to Riemann surfaces, whose study served to introduce the entire subject on a systematic basis, the topic has won as yet but few investigators in America. This is perhaps due to the recentness of most of the developments. While serviceable expositions of much of the theory have been provided by Dehn, Heegard, and Poincaré, there is not a little that is new in this exposition, and one need hardly point out that it is the only work in English to attempt to cover this ground. The subject is basic, important unsolved problems are still open and much machinery is now available. What greater incentive can an adventurous mathematician ask to induce him to read a series of lectures which, like these, are simple to read, are self-contained, and are developed in a progressive logical manner?

ALBERT A. BENNETT.

Précis d'Arithmétique. By J. POIRÉE. Paris, Gauthier-Villars, 1921. 8vo. 5 + 64 pages. Price 7.50 francs.

Author's Preface (translated): "In composing this work, we have not sought to give verbal formulation to the rules of arithmetic, which are found furthermore in all of the treatises. We have been interested particularly in explaining the reason and mechanism of each operation. We have shown how the consideration of sets of objects awakes in us the notion of integral number and how the comparison of magnitudes of the same sort leads to the notion of fraction. The calculation of a fraction to a given degree of approximation permits us to define what is called *limit of a sequence of numbers*. The study of the square root leads us to consider a certain class of these numbers called *irrationals*, of which analysis offers so many examples. If we have spoken of progressions it is because these serve in algebra for the definition of logarithms. We have also shown that progressions enter in certain problems of arithmetic. Finally, the last chapter,

devoted to the examination of the remainder of certain series, will familiarize the reader with the Gaussian notation and will facilitate for him the later study of the theory of numbers which is only sketched here. To close, let us add that different types of arithmetical problems are studied as applications of proportions."

Preface by C. CAMICHEL, Director of the Electrotechnical Institute of the University of Toulouse: "Elementary arithmetic is an excellent introduction to the study of mathematics. There one finds in concrete form models of all types of reasoning from the most simple to the most delicate of analysis. However, this part of mathematics is usually neglected by students. Perhaps they are not entirely to blame. Aiming at extreme rigor and desirous of leaving no difficulty concealed, authors often insist on points whose importance is not well understood by beginners, and which prevent these from seeing the essential parts of the reasoning. Monsieur Poirée has shown how to avoid this danger and has condensed into a few pages the whole object of the classical courses; he has even added a most interesting chapter introducing the reader to the elements of number theory.

"Suppressing all that was not absolutely indispensable, although observing rigor throughout, the author passes rapidly over numeration, the fundamental operations, divisibility, and prime numbers, making the logical order of the theorems clearly evident. Fractions offer him the occasion for introducing and rendering precise the notion of limit, which beginners find so difficult to understand. Similarly, while studying the square root, he introduces, simply and naturally, the irrational numbers. All this is accompanied by numerical examples indispensable to beginners for securing an understanding of the theories.

"Assuredly a treatise so conceived will render the greatest service to all who would understand arithmetic. Like his preceding work, this new book by Monsieur Poirée is a 'real success.'"

Contents—Preface; Chapter I: Numeration, 1–2; II: Fundamental operations, 3–4; III: Tests for divisibility by 2, 5, 4, 3, 9, 11. Casting out nines, 5–6. IV: Greatest common divisor, least common multiple, 7–10; V: Prime numbers, 11–17; VI: Fractions, 18–27; VII: Proportions. Various problems. The metrical system, 28–37; VIII: Square roots. Square roots to the nearest tenth, hundredth, etc. Definition of an irrational number, 38–42; IX: Arithmetic and geometric progressions. Applications, 43–46; X: Introduction to the theory of numbers, 47–62; Table of contents, 63–64.

NOTES ON RECENT PUBLICATIONS.

MORITZ PASCH'S "Die Begriffswelt des Mathematikers in der Vorhalle der Geometrie" has been reprinted from the *Annalen der Philosophie* in pamphlet form (Leipzig, F. Meiner, 1922, 4 + 45 pages; price 50 marks in Germany).

In *Monatshefte für Mathematik und Physik*, Vienna, volume 32, there is a sketch by EMIL MÜLLER (pages 281–293) of the life and work of GUSTAV KOHN, 1859–1921; compare *1922*, 232. There are over fifty titles in the list of his scientific writings.

In the *Quarterly Journal of Pure and Applied Mathematics*, volume 49, number 3, October, 1922, there is an article (pages 226–283) "Abstract definitions of the symmetric and alternating groups and certain other permutation groups" by R. D. CARMICHAEL.

The concluding part (pages 377–524) of the fourth volume of the *Opera Omnia* of Tycho Brahe (1546–1601) was published in 1922 (compare *1920*, 421; *1921*, 314). Volumes I–VI, except for the second part of volume V, have been published since 1912. It will be recalled that this sumptuous publication is being issued by the Danish Society of Language and Literature under the editorship of Professor J. L. E. Dreyer of Oxford.

The mathematical bibliographer and librarian will welcome *Catologo crono-*

logico e alfabetico, per Autori e per Materie, delle Edizioni Hoepli 1872–1922, con Introduzione di Michele Scherillo (Milano, Hoepli, January, 1922, 71 + 404 pages). The exact month in which each one of their five thousand works was published is here indicated; also the dates of different editions. Unfortunately the edition of the *Catologo* was exhausted within six months of its publication.

There is a short memoir on THEODOR REYE (1838–1919), by C. Segre, in *Atti della reale Accademia Nazionale dei Lincei, Rendiconti*, volume 311, pages 268–272, April 2, 1922; on pages 398–404, May 7, there is also a memoir on CAMILE JORDAN (1838–1922), with a list of his principal publications, by L. Bianchi; a brief notice of MAX NOETHER (1844–1921) and his work, by G. Castelnuovo, follows on pages 404–407. Compare this MONTHLY, *1920*, 90; *1922*, 138–139, 233.

The last number of *Nouvelles Annales de Mathématiques*, volume 79, was published in December, 1920; the first number of volume 80 appeared in October, 1922, edited by R. Bricard, H. Villat (the new editor of *Journal de Mathématiques Pures et Appliquées*), and J. Pérès. The periodical is designed, as formerly, to serve candidates in special schools, and those preparing for the examinations of the licence and agrégation. The subscription price is 35 francs a year in the postal union.

In *The Encyclopædia Britannica*, volume 31, 1922, the second of the new volumes, the article on "Mathematics" occupies pages 874–880. The subsections and their authors are as follows: "Mathematical logic and the foundations of mathematics" by JEAN NICOD, agrégé de philosophie; "Theory of numbers" by G. H. HARDY; "Theory of series" by G. H. HARDY; "Theory of functions" by G. H. HARDY; "Geometry" by H. F. BAKER. The article on "Nomography" by R. H. HEZLET occupies pages 1139–1144. In volume 30, 1922, "Aeronautics" and "Ballistics" are extensive articles. The article on "Relativity," volume 33, pages 261–267, is by J. H. JEANS.

Doctor ALFRED ERRERA, secretary of the Belgian Mathematical Society, received the degree of "doctor in physical and mathematical sciences" at the University of Brussels in 1920. His thesis, published at Ixelles, by G. Bothy, in 1921 (66 pages + 1 plate), is entitled *Du Coloriage des Cartes et de quelques Questions d'Analyse Situs*. It contains a fairly complete survey of work done on this problem. In the "Bibliographie" the following Americans are referred to: J. W. ALEXANDER, G. D. BIRKHOFF, H. R. BRAHANA, P. FRANKLIN, N. J. LENNES, ISABEL MADDISON, W. E. STORY, O. VEBLEN. In *Revue de l'Université de Bruxelles*, April–May, 1922, Dr. Errera has a 13-page article entitled, "L'origine et les problèmes de l'analysis situs," and in *Mathesis*, February, 1922, he has a 6-page article, "Analysis situs. Une démonstration du théorème de Petersen."

In this MONTHLY, *1921*, 452, we have discussed the question of the existing copies of Brother Juan Diez's *Sumario Compendioso*, 1556, the first work on mathematics published in the New World. In *Bibliotheca Americana*, catalogue no. 429, published by Maggs Bros., London, in 1922, one of these copies, formerly

in the National Library at Madrid, was offered for sale at £ 120. Of the 103 folios which it contained, 2, 41, and 48 are missing. This seems to be the copy (original vellum with tabs) of which we have already given a reproduction of the title page, *1921*, 11. Maggs Bros. state, on page 48 of their catalogue, that the copy belonging to the British Museum "is non-existent, as we have convinced ourselves by personal research"; nevertheless this is the copy of which there is a rotograph copy at Michigan (*1921*, 452).

In *Journal für die reine und angewandte Mathematik*, volume 152, Heft 1–2, published September 22, 1922, there is a page in memory of K. H. A. SCHWARZ (compare *1922*, 232–233) whose first scientific paper was published in volume 63 when he was a student scarcely twenty years old. This was followed by ten other memoirs in the same journal. He was a member of its editorial committee for volumes 125–151, inclusive. In two papers by RUDOLF STURM, in volume 152, the following problems are discussed: (*a*) The largest tetrahedron with faces of given areas (pages 90–98); (*b*) Through a given point in the plane of two lines, draw a line such that the part cut off by the given lines shall be a minimum; (*c*) Given three planes, and a point or a line, respectively supports of a bundle or a sheaf of planes; to find that one of these planes from which the three given planes cut off the triangle of least area in the solid angle containing the point.

Section 3, December, 1920, of *Publications of the Clark University Library*, volume 6, 1922, contains a "List of degrees granted at Clark University and Clark College 1889–1920." The following 24 men received the degree of Ph.D. in mathematics: J. W. A. YOUNG (1892), W. H. METZLER (1893), T. F. HOLGATE (1893), J. E. HILL (1895), L. W. DOWLING (1895), T. F. NICHOLS (1895), W. G. BULLARD (1896), F. C. FERRY (1898), E. W. RETTGER (1898), J. S. FRENCH (1899), F. B. WILLIAMS (1900), S. E. SLOCUM (1900), H. C. MORENO (1900), H. G. KEPPEL (1901), J. N. VAN DER VRIES (1901), J. N. GATES (1904), R. B. ALLEN (1905), H. L. SLOBIN (1908), R. K. MORLEY (1910), SOLOMON LEFSCHETZ (1911), W. J. MONTGOMERY (1911), J. A. BULLARD (1914), PEYSAH LEYZERAH (1916), SAMUEL ZELDIN (1917). The titles of the theses, and also, in the case of 16, the places of publication, are given. It is noted that a seventeenth was privately printed. The theses of Hill, Dowling, Nichols, and W. G. Bullard were published in *The Mathematical Review* founded by Professor W. E. STORY, of Clark University, and published in Worcester. It has been generally thought that only two numbers of this *Review* were published: I, July, 1896, pages 11–96; II, April, 1896, pages 97–192. It appears, however, that a third number was published in 1899, and that it contained (pages 193–208) the thesis "On the general classification of plane quartic curves" by Professor W. G. Bullard, now of Syracuse University—Section 4, April, 1921, of the *Publications* contains a report of the inauguration of President Atwood of Clark University. Among the published greetings are those of A. P. WILLS, professor of mathematical physics at Columbia University, and of FRANK MORLEY, professor of mathematics in the Johns Hopkins University.

PROBLEMS AND SOLUTIONS.

EDITED BY B. F. FINKEL, OTTO DUNKEL, AND NORMAN ANNING.

Send all communications about Problems and Solutions to B. F. FINKEL, Springfield, Mo.

PROBLEMS FOR SOLUTION.

[N. B. Problems containing results believed to be new, or extensions of old results, are especially sought. The editorial work would be greatly facilitated if, on sending in problems, proposers would also enclose any solutions or information that will assist the editors in checking the statements. In general, problems in well-known text-books, or results found in readily accessible sources, will not be proposed as problems for solution in the MONTHLY. In so far as possible however, the editors will be glad to assist the members of the Association with their difficulties in the solution of such problems.]

3004. Proposed by W. B. FORD, University of Michigan.

Is there a plane curve such that two tangents whose lengths are in the constant ratio $n : 1$ $(n \neq 1)$ may be drawn to it from any point in its plane? If so, discuss its properties.

3005. Proposed by F. D. MURNAGHAN, Johns Hopkins University.

Given the recurrence formulæ: $x_{n+1} = x_n(x_n + y_n)$, $y_{n+1} = y_n(2x_n + y_n)$ with the initial values $x_1 = 1$, $y_1 = 1$, it is desired to find a good approximation to $\dfrac{x_n}{2x_n + y_n}$ for large values of n. In particular, for values of n from $n = 20$ to $n = 30$.

Note.—This question arose in connection with a problem in genetics.

3006. Proposed by S. A. COREY, Des Moines, Iowa.

The formula

$$m\int f'(mx)dx = x\{c + f'(0) + f'(x) + f'(2x) + f'(3x) + f'(4x) + \cdots + f'[(m-1)x] + \tfrac{1}{1440}\{965f'(mx) - 462f'[(m+1)x] + 336f'[(m+2)x] - 146f'[(m+3)x] + 27f'[(m+4)x]\},$$

gives the exact value of the integral when the fifth and all higher derivatives are zero. Find an expression for the remainder term in the general case.

3007. Proposed by NORMAN ANNING, University of Michigan.

Given two opaque spheres, radii r_1 and r_2, at a fixed distance $d(> r_1 + r_2)$ apart; locate the points on either from which the maximum surface of the other can be seen.

3008. Proposed by P. R. RIDER, Washington University.

The altitude of a right circular cone is a, the radius of its base is b, and its slant height is c. A string is wrapped n times about the cone, starting at the vertex and ending at the base, in such a manner that for any complete circuit the vertical rise (the cone being supposed to rest on its base) is the same. A bird at the vertex takes the end of the string in its beak and flies around the cone, unwinding the string, keeping it taut and always tangent to the curve of the string as it lies around the cone. Find an expression for the distance that the bird has flown when the string is completely unwound, (*a*) if it starts at the vertex, (*b*) if it starts at the base.

3009. Proposed by J. G. COFFIN, New York City.

Rectangular pieces of cardboard of the same dimensions are piled so that they overhang to the greatest extent possible; what curve do the edges touch? how great a distance between first and last piece can be obtained? and what are the properties of the material volume thus produced?

It is assumed that the solver will be led to consider an infinite number of infinitely thin layers.

3010. Proposed by F. D. MURNAGHAN, Johns Hopkins University.

Find an expression for the volume of the pedal tetrahedron, with respect to the tetrahedron of reference, of a point whose perpendicular distances from the sides are (x_1, x_2, x_3, x_4); from this expression show that the locus of points the feet of whose perpendiculars on the faces of the

tetrahedron of reference are coplanar is Steiner's cubic surface,

$$\frac{A_1}{x_1}+\frac{A_2}{x_2}+\frac{A_3}{x_3}+\frac{A_4}{x_4}=0,$$

where the A's are the areas of the faces of the tetrahedron of reference.

SOLUTIONS.

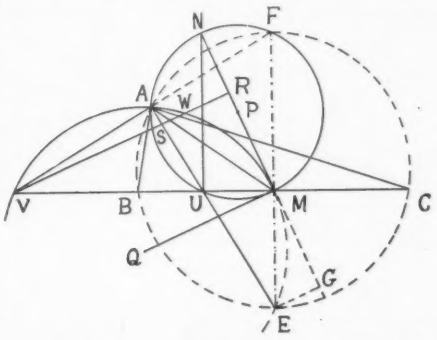
2903 [1921, 277]. Proposed by A. A. BENNETT, University of Texas.

Given the base of a triangle in position and length, the length of the median to the base, and the difference of the base angles; find a simple ruler and compasses construction for the triangle.

Two Solutions by Nathan Altshiller-Court, University of Oklahoma.

I. Analysis. Let ABC be the required triangle with $\angle B \geqq \angle C$; let AM be the given median, and let AU and AV be the interior and exterior bisectors of the angle A. From the triangle ABU we have: $\angle AUM = \angle B + \frac{1}{2}\angle A = \frac{1}{2}\angle B + \frac{1}{2}(\angle A + \angle B)$. But $\frac{1}{2}(\angle A + \angle B + \angle C) = 90°$; hence $\angle AUM = 90° + \frac{1}{2}(\angle B - \angle C) = 90° + d$, where $2d$ is the given difference of the two base angles of the triangle. Thus the given median AM subtends at U an angle of known magnitude; therefore, the point U lies on a circle (P), which may readily be constructed.

From the triangle AUV we have: $\angle AVU = \angle AUM - \angle VAU = 90° + d - 90° = d$. Thus again the given median AM subtends at the point V an angle of known magnitude, and, therefore, V lies on a circle (Q), which may readily be constructed.



Let R be the foot of the perpendicular dropped from the point V upon the diameter MPN of the circle (P). The two right triangles MRV and MNU, having the acute angle M in common, are similar; hence, $MR \cdot MN = MU \cdot MV$. Now the product of the last two segments is known. In fact, the points U, V divide the base BC harmonically, and since M is the mid-point of BC, we have:[1] $MU \cdot MV = \overline{MB}^2 = a^2$, if $2a$ denotes the given length of the base BC of the triangle ABC. Thus $MR \cdot MN = a^2$. The diameter MN is known; hence this equality determines the segment MR and the point R, so that the perpendicular RV may be constructed. Thus the point V lies on the circle (Q) and on a given line RV. (The reader who is familiar with the method of inversion will readily notice that the perpendicular RV is the inverse of the circle (P) with respect to the center of inversion M, the power of inversion being equal to a^2.)

Construction. Let $2a$, m, $2d$ denote, respectively, the magnitudes of the given base, of the given median, and of the given difference of the base angles.

Draw a right angle PMQ, and a line MA so that $\angle AMP = d$. Lay off $MA = m$, and let the perpendicular bisector of MA meet the lines MP, MQ in the points P, Q, respectively. Draw the two circles having P and Q for centers, and PA, QA for radii. On the diameter MPN lay off the segment MR such that $MR \cdot MN = a^2$. At the point R erect a perpendicular to MN, meeting the circle (Q) beyond A in the point V (see discussion below). On the line MV lay off $MB = MC = a$. The triangle ABC satisfies the conditions of the problem.

Proof. The triangle ABC has the required base and the required median by construction. It remains to show that $\angle B - \angle C = 2d$.

The line MN being drawn perpendicular to QM touches the circle (Q) at M. The inscribed angle AVM is equal to the angle formed by the chord AM and the tangent to (Q) at M. But $\angle AMN = d$, by construction; hence, $\angle AVM = d$. In a similar way, it may be shown that the angle AUM, where U is the second point of intersection of MV with circle (P), is equal to $90° + d$. Consequently, the $\angle UAV = \angle AUM - \angle AVM$ is a right angle.

[1] John W. Russell, *A Sequel to Elementary Geometry*, Oxford, 1913, p. 54.

By construction $MR \cdot MN = a^2$, and from the similar right triangles MRV and MNU we have $MU \cdot NV = MN \cdot MR$; hence $MU \cdot MV = a^2$. Therefore, the four points B, C, U, V are harmonic.[1] On the other hand UAV is a right angle, as shown above; consequently, the lines AU, AV are the bisectors of the angle BAC.[2]

From the triangle ABU we have:

$$\angle AUM = \angle UAB + \angle ABC. \tag{1}$$

Since $\angle AUM = 90° + d = \frac{1}{2}(\angle A + \angle B + \angle C) + d$, and $\angle UAB = \frac{1}{2} \angle A$, (1) becomes $\frac{1}{2}(\angle A + \angle B + \angle C) + d = \frac{1}{2} \angle A + \angle B$; whence, $\frac{1}{2}(\angle B - \angle C) = d$.

Discussion. The difference $2d$ of the two base angles must necessarily be smaller than 180°. If this condition is fulfilled, the two circles (P), (Q) can always be constructed, and the solution of the problem depends upon the finding of the point V.

In the case of the figure, the perpendicular VR meets the circle (Q) in another point W, and the two points V, W lie on opposite sides of the median AM. However, the line MW, lying between AM and MN, cannot be taken for the line of the base of the required triangle, because the triangle constructed on MW would have for the difference of its base angles not $2d$, but its supplement. If S is the point of intersection of VR with (P), we have from the right triangle MNS, $MS^2 = MN \cdot MR = a^2$; hence, $MS = a$. Thus the two points of intersection of (Q) with VR will lie on opposite sides of AM, when $a < m$. The problem will have one solution. The angle at A will be acute.

If $a = m$, the triangle ABC will have a right angle at A.

If $a > m$, and MR is smaller than the radius of (Q), the perpendicular VR will meet (Q) in two points, which will lie on the same side of AM. The problem will have two solutions, if $d < 45°$. The angle at A will be obtuse. If MR is equal to the radius of (Q), the two solutions will coincide. If $d \geqq 45°$ the two points will lie between A and M and there will be no solution.

If $a > m$, and MR is larger than the radius of (Q), the problem will have no solutions.

If A' is the symmetric of A with respect to the perpendicular bisector of the base BC, the triangle $A'BC$ satisfies the conditions of the problem.

If the base is given in position, the above construction determines the angle between the base and the median, and the construction of the required triangle is easily completed.

If, in the analysis of the problem, instead of dropping from the point V of (Q) a perpendicular upon the diameter MN of (P), we had dropped from the point U of (P) a perpendicular upon MQ, the rôle of the two circles (P), (Q) in the construction of the triangle ABC would have been interchanged.

II. Let ABC be the required triangle, and EF the diameter of the circumcircle perpendicular to the base BC. The angles ABC, ACB, AEF have for their respective measures half the arcs AFC, AB, AF. But arc AFC − arc AB = 2 arc AF, hence $\angle AEF = \frac{1}{2}(\angle B - \angle C) = d$, if $2d$ is the given difference of the base angles of the triangle.

From the right triangle AEF we have $\angle AFE = 90° - d$. Thus the given median AM subtends at the points E and F angles of given magnitude; hence, the points E, F lie on two circles (Q), (P), which may readily be constructed. Let G be the foot of the perpendicular EG from E to the diameter MPN of the circle (P). From the similar right triangles MEG and MFN, we have: $MG \cdot MN = ME \cdot MF$. But from the circumcircle of ABC we have: $ME \cdot MF = MB \cdot MC = a^2$; if $2a$ is the given length of the base BC, hence $MG \cdot MN = a^2$. Since MN is known, the length MG, and therefore also the point G, may be determined. Thus the point E lies on the circle (Q) and on the perpendicular EG to a given line MN at a given point G. The base BC of the required triangle is perpendicular to the known line ME.

It is easy to see that the lines AE, AF are the bisectors of the angle A; that is, they are identical with the lines AU, AV considered in the first solution, and that the circles (P), (Q) are the same as considered above.

III. Discussion by Otto Dunkel, Washington University.

In a triangle ABC with BC fixed and equal to $2a$, let θ be the angle at C and $\theta + 2d$ the angle at B. Then the rays BA and CA will describe equal pencils and the locus of A will be a conic passing through B and C, there being no self-corresponding ray. When $\theta = 0$, BA becomes tangent at B, making the angle $2d$ with BC; when $\theta = -2d$, CA is a tangent at C. Since these

[1] Russell, *loc. cit.*, p. 55.

[2] Russell, *loc. cit.*, p. 58.

two tangents are parallel, BC is a diameter and its mid-point M is the center. When $\theta = 90° - d$, CA and BA are parallel, and when $\theta = -d$, they are parallel again. Hence the conic is a rectangular hyperbola, its asymptotes making angles $90° - d$ and $-d$ with MB. Since B is regarded as the larger of the two angles it is only necessary to consider that part of the branch through B which lies above MB, for this gives the extreme variations of B from $2d$ to $90° + d$. If I is the vertex of the hyperbola on the branch through B, and if the x-axis is taken along MI and the origin at M, the equation of the curve is $x^2 - y^2 = MI^2$. As MB is equal to a, the coördinates of B are $a\cos(45° - d)$ and $a\sin(45° - d)$, and therefore $MI = \sqrt{\sin 2d}$.

Now if m is the length of the median the circle with center at M and radius m will cut this part of the hyperbola in the desired point A. This leads to the following cases:

$$\begin{array}{rll} m < a\sqrt{\sin 2d} & & \text{no solution} \\ m = a\sqrt{\sin 2d} \text{ and } & \begin{cases} d < 45° \\ d \geqq 45° \end{cases} & \begin{array}{l} \text{one solution} \\ \text{no solution} \end{array} \\ a > m > a\sqrt{\sin 2d} \text{ and } & \begin{cases} d < 45° \\ d \geqq 45° \end{cases} & \begin{array}{l} \text{two solutions} \\ \text{no solution} \end{array} \\ m \geqq a & & \text{one solution.} \end{array}$$

The abscissa of the vertex is found to be $\sqrt{(m^2 + a^2 \sin 2d)/2}$ and this may be used for a construction, but such a construction would not be nearly as elegant as the author's.

In the figure above $MN = m/\cos d$, $MR = (a^2 \cos d)/m$, and $MQ = m/(2 \sin d)$. Therefore $MR \gtreqless MQ$ according as $a\sqrt{\sin 2d} \gtreqless m$.

Also solved by J. F. Howard, Arthur Pelletier, and F. L. Wilmer.

2917 [1921, 327].

A parabola is rolled upon a fixed right line. Find the locus of (*a*) its vertex and (*b*) its focus.

Solution by J. B. Reynolds, Lehigh University.

If the parabola with vertex V and focus F has contact at C with the fixed line and if FV cuts that line in S, then from well-known properties of the parabola the triangle SFC is isosceles and the tangent at the vertex V passes through M, the middle point of SC. Hence MF is perpendicular to SC. Since C is the instantaneous center of rotation, the tangent to the locus of F is perpendicular to FC. Let this tangent cut SC in T and set $\tau = \angle MTF = \angle VFM$, $MF = y$, $VF = a$. Then

$$y = a \sec \tau, \tag{1}$$

which is characteristic of the catenary.

To deduce from (1) the ordinary equation of the curve, we have $dy = a \sec \tau \tan \tau \, d\tau$ and $dy = \tan \tau \, dx$ and hence $dx = a \sec \tau \, d\tau$ or

$$x = a \log(\sec \tau + \tan \tau) = -a \log(\sec \tau - \tan \tau). \tag{2}$$

By eliminating $\tan \tau$ from the two expressions for x and using (1) we find

$$2y = a(e^{x/a} + e^{-x/a}). \tag{3}$$

In order to obtain the parametric equations of the path of V, we have merely to subtract from the right hand sides of (2) and (1) $a \sin \tau$ and $a \cos \tau$, respectively, giving for the required locus

$$\begin{aligned} x &= a[\log(\sec \tau + \tan \tau) - \sin \tau], \\ y &= a \sin \tau \tan \tau. \end{aligned}$$

Also solved by George Agins, T. L. Bennett, T. M. Blakslee and William Hoover.

2919 [1921, 327]. Proposed by V. M. SPUNAR, Chicago, Ill.

An equilateral hyperbola which touches a conic and is concentric with it is called a hyperbolic tangent to the conic. Being given two hyperbolic tangents to a conic, the arc of any third hyperbolic tangent which is intercepted by the first two subtends a constant angle at either focus of the given conic.

I. Solution by William Hoover, Columbus, Ohio.

This problem has added interest on account of its history. In *Liouville's Journal*, volume 13, 1848, p. 209, a letter of William Roberts to the journal dated Dublin, November 10, 1847, appears in which he obtains a large number of theorems by use of the transformation in polar coördinates

$$r = R^n, \qquad \omega = n\Omega, \tag{1}$$

where n is any number. This transformation is conformal, he points out, for $rd\omega/dr = Rd\Omega/dR$ If we apply (1) to the straight lines of the plane,

$$r\cos(\omega - \alpha) = p, \tag{2}$$

we obtain the set of curves,

$$R^n\cos(n\Omega - \alpha) = p, \tag{3}$$

which, for a given α, is a family of curves having an axis inclined at the angle α/n to the initial line. If $p = 0$ the curves in (2) and (3) are straight lines through the pole. If $p \neq 0$ and $n = 2$, (3) is a system of equilateral hyperbolas with centers at the pole; if $n = 1/2$, confocal parabolas. Since (1) is a conformal transformation, the curves of the two families given by $\alpha = \alpha_1$ and $\alpha = \alpha_2$ cut each other under the constant angle $\alpha_2 - \alpha_1$, which is n times the angle between their respective axes.

The transformation (1) for $n = 2$ applied to the central conics

$$r(1 - e\cos\omega) = l \tag{4}$$

gives the central conics $R^2(1 - e\cos 2\Omega) = l$, for which the focus of the former goes into the center of the latter. For convenience of statement Roberts introduces the term hyperbolic tangent defined in the problem. Now it is known that, if the two points in which two fixed tangents to a conic are cut by a third tangent be joined by straight lines to a focus, the latter two lines include a constant angle.[1] If the conic is central, this theorem goes over by (1) and the facts above into the theorem of the problem.

Another theorem given by Roberts is as follows: Being given a central conic and a concentric equilateral hyperbola, if from any point on the latter are drawn two hyperbolic tangents to the first conic, the concentric equilateral hyperbola passing through the two points of contact will pass through a fixed point. This results easily by passing from well known theorems regarding poles and polars of a conic by (1) for $n = 2$ to the corresponding theorems of the transformed figure.

He also states that a system of homofocal conics transforms into a system of homofocal conics.

II. Remarks by Otto Dunkel, Washington University.

It should be observed that the real proof in the above consists in transforming the theorem to be proved into known theorems, or by showing that the reverse process covers all cases. Either procedure is easy here.

The transformation used here is merely a special case of transformation in polar coördinates which satisfy the condition of conformality:

$$\frac{r}{R}\frac{\partial\omega}{\partial\Omega} = \frac{\partial r}{\partial R}, \qquad rR\frac{\partial\omega}{\partial R} = -\frac{\partial r}{\partial\Omega}.$$

2920 [1921, 392]. Proposed by N. P. Pandya, Sojitra, India.

Construct a triangle, having given the base, the angle between the base and the median on it, and the difference of the remaining two sides.

I. Solution by Arthur Pelletier, Montreal, Canada.

Let $c = AB$ be the given base, d the difference between the remaining two sides, $d < c$, DF the position of the median making the acute angle α with DB. Describe a circle with center A and radius d, and construct the point B' symmetrical to B with respect to DF. Pass a circle through B and B' tangent to the above circle (a well-known problem). The center C of this circle is evidently the remaining vertex of the required triangle.

[1] See Reye, *Die Geometrie der Lage*, 1909, part 1, p. 165, where the theorem is attributed to Poncelet.

II. Notes by Otto Dunkel, Washington University.

Two circles can be passed through B and B' tangent to the circle with center A, one tangent externally and the other tangent internally. In this particular problem the center C' of the second tangent circle gives a triangle BAC' which may be easily shown to be congruent to the triangle ABC.

The point B' lies on a circle with AB as a diameter and $\angle BAB' = \alpha$. From this it follows at once that the construction is not possible if $c \cos \alpha \leqq d$.

The tangent circles C and C' may be constructed as follows: Pass any circle through B and B' cutting the circle A in two points and let their common chord cut BB' in I. Then the circle constructed with IA as a diameter cuts the circle A in G and G' which are the points of tangency of the required circles.

Also solved by A. R. Nauer, J. B. Reynolds, Marcus Skarstedt, J. H. Weaver and F. L. Wilmer.

2927 [1921, 393]. Proposed by Philip Franklin, Harvard University.

Prove that the only positive integral values greater than unity which satisfy the equation $3^x - 2^y = \pm 1$ are $x = 2$, $y = 3$. (Cf. Carmichael, *Diophantine Analysis*, 1915, p. 116, exercise 69.)

Solution by the Proposer.

(*a*) The equation, $3^x - 2^y = 1$, implies

$$2^y + 1 = (4 - 1)^x.$$

Since y is greater than one, the left and consequently the right member of this equation must be congruent to 1, modulo 4, and x must be even. We may therefore write

$$2^y + 1 = (2m + 1)^2 = 4m(m + 1) + 1.$$

This shows that m must be unity, since otherwise $m(m + 1)$ would contain an odd factor. Hence $(x, y) = (2, 3)$ is the only pair of integers each greater than unity satisfying (*a*).

(*b*) The equation, $3^x - 2^y = -1$, implies

$$3^x + 1 = (3 - 1)^y.$$

By taking the remainders, modulo 3, we show that y is even. We may thus write

$$3^x + 1 = (3m \pm 1)^2 = 3m(3m \pm 2) + 1.$$

The only value of m which makes $m(3m \pm 2)$ a power of 3 is unity, and we must take the minus sign. As this requires x to be unity, it does not lead to any further solutions of the kind specified.

Also solved by F. L. Wilmer.

2929 [1921, 466]. Proposed by R. E. Gaines, University of Richmond.

Denote by A, O and B, respectively, the points $(-1, 0)$, $(0, 0)$ and $(1, 0)$ on the curve $y^2 = x^3 - x$, and let P be a variable point on the curve. Let PA and PO meet the curve again in Q and R, respectively, and let BQ and BP meet AR in M and N, respectively. Prove that QN is perpendicular to PM.

Solution by the Proposer.

The coördinates of P and Q must satisfy the equation of the curve and $\lambda y = x + 1$ and hence also the equation $y - \lambda x(x - 1) = 0$. If we take B as origin, the last two equations become $y - \lambda x' - \lambda x'^2 = 0$ and $\lambda y - x' = 2$. Making the first equation homogeneous by aid of the second we have

$$\lambda(y^2 - x'^2) - (\lambda^2 + 1)x'y = 0$$

as the equation of BQ and BP and it follows at once that they are perpendicular.

In a similar manner, by finding the intersections of RP with the curve and taking A as the origin we show that AR and AP are perpendicular. In the triangle MNP the two altitudes MB and PA meet in Q and hence NQ must be the third altitude, *i.e.*, perpendicular to MP.

Also solved by E. F. ALLEN, T. M. BLAKSLEE, R. M. MATHEWS, E. J. OGLESBY, ARTHUR PELLETIER and J. B. REYNOLDS.

2930 [1921, 467]. Proposed by R. E. GAINES, University of Richmond.

If in reducing p/q (p and q integers, $q > p$) to a decimal the remainder $q - p$ ever appears, then the fraction will give a repeating decimal the number of digits in whose repetend will be exactly twice the number of digits in the quotient already obtained, and the remaining digits may, without further division, be obtained by subtracting the quotient already found from a succession of 9's.

SOLUTION BY E. M. BERRY, Purdue University.

Suppose $\frac{p}{q} = \cdot d_1 d_2 d_3 \cdots d_n + \frac{q-p}{q} 10^{-n}$, where d_i is the digit in the ith decimal place. Then

$$\frac{q-p}{q} = 1 - \frac{p}{q} = 1 - \cdot d_1 d_2 d_3 \cdots d_n - 10^{-n} + \frac{p}{q} 10^{-n}$$

$$= \cdot (9-d_1)(9-d_2)(9-d_3) \cdots (9-d_n) + \frac{p}{q} 10^{-n},$$

since $1 - 10^{-n} = .999 \cdots$ to n decimal places. Thus

$$\frac{p}{q} = \cdot d_1 d_2 d_3 \cdots d_n (9-d_1)(9-d_2)(9-d_3) \cdots (9-d_n) + \frac{p}{q} 10^{-2n}$$

where $(9 - d_i)$ is the digit in the $(n + i)$th decimal place. Hence the theorem is proved.

Also solved by MICHAEL GOLDBERG, R. M. MATHEWS, ARTHUR PELLETIER, and the PROPOSER.

2931 [1921, 467]. Proposed by R. C. ARCHIBALD, Brown University.

From the equality $\sec(\pi/14) + \sec(3\pi/14) - \sec(5\pi/14) = 0$ find a relation between the lengths of the side and diagonals of a regular inscribed heptagon.

SOLUTION BY J. B. REYNOLDS, Lehigh University.

Let z be the side of the regular heptagon inscribed in a circle of radius r, and let x and y be the longer and shorter diagonals. Then $x = 2r \sin 3\pi/7 = 2r \cos \pi/14$ or $\sec \pi/14 = 2r/x$. Similarly, $\sec 3\pi/14 = 2r/y$, $\sec 5\pi/14 = 2r/z$. Substituting these values in the given equality, we have

$$\frac{1}{z} = \frac{1}{x} + \frac{1}{y}.$$

Hence the side is one half the harmonic mean of the two diagonals. If x and y are taken as two sides of a triangle including an angle of 120°, z is the length of the bisector of this angle.

Also solved by ARTHUR PELLETIER and T. M. BLAKSLEE.

NOTES AND NEWS.

It is hoped that readers of the MONTHLY will coöperate in contributing to the general interest of this department by sending items to R. W. BURGESS, Brown University, Providence, R. I.

Assistant Professor C. E. WILDER, of Northwestern University, has been appointed assistant professor of mathematics at Dartmouth College.

Assistant Professor C. E. MELVILLE, of Clark University, has been promoted to a full professorship.

Miss MARTHA F. CHADBOURNE has been appointed instructor of mathematics at Wheaton College, Norton, Mass.

Dr. C. R. ADAMS, of Harvard University, has been appointed instructor of mathematics at Brown University for the year 1923–1924.

At Trinity College, Hartford, Conn., Mr. F. J. BURKETT, of the University of Pittsburgh, has been appointed assistant professor of mathematics, and Assistant Professor H. M. DADOURIAN has been promoted to an associate professorship of physics.

Assistant Professor MARY E. WELLS, of Vassar College, has been promoted to an associate professorship of mathematics.

At Princeton University, Assistant Professor J. H. M. WEDDERBURN has been promoted to an associate professorship of mathematics, and Dr. EINAR HILLE and Mr. G. B. BRIGGS have been appointed instructors.

Assistant Professor C. C. Bramble, of the U. S. Naval Academy, has been promoted to an associate professorship of mathematics.

Mr. F. H. MURRAY (see *1921*, 191) has been appointed instructor of mathematics at the University of West Virginia.

Mr. F. L. BROWN, of Northwestern University, has been appointed assistant professor of mathematics at the University of Virginia.

At the University of North Carolina, Assistant Professor E. T. BROWNE, of Trinity College, Hartford, Conn., has been appointed assistant professor of mathematics, and Assistant Professor J. W. LASLEY, JR. has been promoted to an associate professorship.

Professor A. T. DELURY, of the University of Toronto, has been appointed Dean of the Faculty of Arts.

At Queen's University, Kingston, Ontario, Associate Professor C. F. GUMMER. has been promoted to a full professorship, and Dr. NORMAN MILLER has been promoted to an associate professorship.

Mr. WALTER DENSTON, formerly of the Imperial Naval Engineering College, Kronstadt, Russia, has been appointed assistant professor of mathematics at Kenyon College.

Associate Professor LOUIS BRAND, of the University of Cincinnati, has been promoted to a full professorship of mathematics.

At Purdue University, Professor WILLIAM MARSHALL has been appointed head of the department of mathematics. Assistant Professor R. B. STONE, who is also Registrar, has been promoted to an associate professorship, and Dr. W. E. EDINGTON, Mr. W. H. LYONS and Mr. W. J. WAGNER have been appointed instructors.

Dr. C. P. SOUSLEY is professor and head of the department of mathematics at Rose Polytechnic Institute, Terre Haute (not associate professor, as listed in the Register).

At Indiana University, Associate Professor U. S. HANNA has been promoted to a full professorship, and Assistant Professor CORA B. HENNEL to an associate professorship of mathematics.

At Northwestern University, Professor F. E. WOOD of the Michigan Agricultural College has been appointed assistant professor of mathematics, and Mr. R. L. JACKSON and Mr. J. D. VASS have been appointed instructors.

Assistant Professor W. L. MISER, of the Armour Institute of Technology, has been promoted to an associate professorship of mathematics.

Dr. MAYME I. LOGSDON has been appointed instructor of mathematics at the University of Chicago.

Dr. C. C. WYLIE is associate in astronomy and acting head of the department at the University of Illinois.

At the University of Iowa, Dr. W. H. WILSON has been appointed assistant professor of mathematics, Dr. ROSCOE WOODS has been appointed associate, and Mr. R. E. KENNON has been appointed instructor.

At Washington University, St. Louis, Assistant Professor P. R. RIDER has been promoted to an associate professorship of mathematics, and Mr. THEODORE DOLL, of Northwestern University, has been appointed assistant professor.

Mr. V. B. HINSCH, of the Missouri School of Mines, has been promoted to an assistant professorship of mathematics.

Professor E. B. STOUFFER, of the University of Kansas, has been appointed acting dean of the Graduate School.

Assistant Professor E. D. MEACHEM, of the University of Oklahoma, has been promoted to an associate professorship of mathematics.

Professor J. E. BURNAM, of Simmons College, Abilene, Texas, has leave of absence for the present college year and is studying as a fellow at the University of Texas.

Mr. R. G. LUBBEN has been appointed instructor of mathematics at the University of Texas.

Professor J. N. DONOHUE, of Notre Dame University, has been appointed professor of mathematics at Columbia University, Portland, Oregon.

Dr. NINA M. ALDERTON has been promoted to an assistant professorship of mathematics at Mills College (California).

At the University of California, Dr. B. C. WONG, Dr. ELSIE MCFARLAND, and, at the Southern Branch, Mr. PAUL DAUS, have been appointed instructors of mathematics.

Professor A. SOMMERFELD delivered a course of lectures at the Bureau of Standards, Washington, early in March, 1923, on the quantum theory and related subjects.

Professor C. KOSTKA of Insterburg, died Dec. 28, 1921, at the age of seventy-five years.

Professor ERNEST LEBON, of the Lycée Charlemagne, Paris, died February 12, 1922, in his seventy-sixth year.

Dr. A. R. WILLIS, of the department of mathematical physics at the Royal College of Science, died June 23, 1922, at the age of seventy-two years.

The Fall Meeting of the New York Section of the Association of Teachers of Mathematics in the Middle States and Maryland was held at Hunter College, New York City, December 15, 1922. The program included "The great problem of algebra" by D. E. SMITH; a debate on the subject "Resolved, that the mathematics of the first year of the New York City standard high-school course should be confined and held to algebra," in which E. H. KOCH and FLETCHER DURRELL

spoke for the affirmative; and "A proposed new course for senior high-school mathematics" by G. R. MIRICK.

The following awards of Nobel prizes in physics have been announced: for 1921, to Professor ALBERT EINSTEIN, for his theory of relativity and his general work in physics; for 1922, to Professor NIELS BOHR, for his researches in the structure of the atom and in radiation.

The Paris Academy of Sciences has awarded its Lalande Medal to Professor H. N. RUSSELL, of Princeton University, and its Janssen Medal to Dr. CARL STORMER, professor of pure mathematics at the University of Christiania, for his work on the aurora borealis.

The Royal Society of London has awarded its Sylvester Medal to Professor TULLIO LEVI CIVITA, for his researches in geometry and mechanics.

HANS BATTERMANN, professor of astronomy at the University of Königsberg and director of the University Observatory, died June 15, 1922. He was born in 1860. "In addition to valuable work with the transit circle, Professor Battermann carried out valuable investigations on the heliometer, and made accurate determination of the solar parallax from a long series of lunar occultations."

EDMOND HERBERT GROVE HILLS, author of many papers on astronomical and allied subjects, died October 2, 1922. He was born at Winchester, England, August 1, 1864. After a course at the Royal Military Academy, Woolwich, he received a commission in the Royal Military Engineers in 1884. In 1893 he was elected a fellow of the Royal Astronomical Society (president, 1913–1915) and he took part in various eclipse expeditions, the last that of 1914 in Kieff, Russia, whence he was recalled for military service. He became a colonel in 1914 and a brigadier-general in 1918. His notable work in the topographical section of the war office was recognized by the order Commander of St. Michael and St. George, conferred on him in 1902. He was elected a fellow of the Royal Society in 1911, and made a Commander of the Order of the British Empire in 1919.

CHARLES MICHIE SMITH, government astronomer at Madras, 1891–1911, and director of the Kadaikánal and Madras observatories 1899–1911, died September 27, 1922. He was born in Keig, Aberdeenshire, Scotland, July 13, 1854, was educated at Aberdeen and Edinburgh Universities, and was appointed professor of physical science in Madras Christian College in 1877. Apart from publications of the observatories of which he was a director (these included, for example, *New Madras General Catalogue of 5303 Stars*, 1892, 314 pages) most of his papers appeared in the *Proceedings* and *Transactions of the Royal Society of Edinburgh*. He was made a companion of the Order of the Indian Empire in 1910.

HENRY TRESAWNA GERRANS, fellow, vice-provost, and lecturer in mathematics at Worcester College, Oxford, died June 20, 1921. He was born at Plymouth, England, August 23, 1858. As a student at Christ Church, Oxford, he won first class in mathematics in the "mods." and "greats." He was appointed a fellow of Worcester in 1882 and later occupied numerous executive positions in connection with the University including that of being a delegate of the Oxford University Press since 1896. He was a member of many mathematical and scientific societies. As editor of the fifth edition of the second volume of G. M. Minchin's *Treatise on Statics* (Oxford, 1915) he made large additions to the work in connection with problems for solution. "He was a musician, and as much at home in German as in English. Thirty years ago his rooms on Sunday evening used to be crowded with foreigners, with all of whom, it was said, he could converse fluently." At one time, "he considered that continental holiday ill-spent in which he did not master some unlearnt language." A portrait and sketch of Mr. Gerrans appeared in *The Periodical*, Oxford, September, 1921, page 95.

CARGILL GILSTON KNOTT, lecturer on applied mathematics at the University of Edinburgh since 1892, and general secretary of the Royal Society of Edinburgh, died October 26, 1922. He was born at Penicuik, Scotland, June 30, 1856, and became a graduate of the University of Edinburgh. During 1883–1891 he was professor of physics at the University of Tokyo, Japan, and was decorated with the Fourth Order of the Rising Sun in 1891. He was honorary secretary of the Napier Tercentenary Celebration, Edinburgh, 1914, and editor of its sumptuous *Memorial Volume*, London, 1915, which contained papers by more than a score of contributors including the American authors, F. Cajori, A. Martin, and D. E. Smith. He was the editor of *Edinburgh's Place in Scientific Progress*, Edinburgh, 1921, and he wrote the chapter on "Mathematics and natural philosophy" (compare *1921*, 456). His third edition of Kelland and Tait's *Quaternions*, London, 1904, and his elaborate *Life and Scientific Work of Peter Guthrie Tait*, Cambridge, 1911, are well known.

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